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Abstract

The report contains a review of basic models and mathematical tools used in economic analysis of environmental regulation problems. It starts with presentation of basic models of capital accumulation, resource depletion, pollution accumulation and population growth, as well as construction of utility functions. Then the one-state variable model is discussed in details. The basic mathematical methods used consist of application of the maximum principle and phase plane analysis of the differential equations obtained as the necessary condition of optimality. A summary of basic results connected with these methods is given in appendices.

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Chapter 1

Introduction

Usually there is a tradeoff between economic development and preservation of the environment. The difficulty in economic tackling of the problem arises because of difficulty in estimating the values of wildlands and its benefits to the human life, and moreover in prediction of the future values and benefits, like e.g. from preserving natural biota or botanic specimen. This is also connected with irreversibility of many development projects when their influence on the environment is considered. It was probably John Stuart Mill [50] who first emphasized the importance of the environment for the quality of life. More elaborated views on the economics of the problem, influencing further studies, were, however, formulated only in the 1960s, with the works of Krutilla [39] and Boulding [6]. These issues have further been intensively discussed in the economic literature and an extensive list of literature on this subjects can be found in survey papers, like [22] or [13].

This report is concerned with the use of *mathematical models* in the study of economic environmental issues, and specifically, the *dynamic models.* There is quite a tradition that the dynamic economic problems involving the dynamic models are coped with by using the optimal control theory. This idea can be traced to early applications of variational methods in [20], [58] or [32], but mainly arose in the second half of the 1960s where newly developed control theory tools like the maximum principle and dynamic programming went into use. In the 1970s many books presenting this approach appeared, and particularly the very influential book by Arrow and Kurz [1]. See also the historical remarks in [21]. For the discussion of static models and methods of its analysis see e.g. [4] or [64].

There is quite a handful of books dealing with the economics of environment, like [9], [17], [64], [35], [21], [12], [4]. But the area discussed in this report includes also many vast subjects with independent developing literature. This is cited in Chapter 2 where these subjects are shortly presented.

The aim of this report is twofold. One is to review the area and to introduce the reader, possibly not deeply acquainted with it, with models, methods and problems spotted there. This is mainly expected to be achieved in Chapter 2. The second is connected with a deeper mathematical analysis of the models and methods. This subject is included in Chapter 3, where the one state variable model is extensively analysed. This chapter comprises also nonstandard analysis of the phase planes with a stationary point on a boundary, which has been found either to be presented in error or missing in the literature. Appendices A and B contain short presentations of the basic theoretical tools used in the analysis.

The limitations connected with the presently available time resources drove us to the idea of presenting the results at the present stage of research. However, the survey of problems is by no means completed here and some further research problems to be explored are mentioned in Chapter 4. These are expected to be the subject of future activity.

Chapter 2

Review of Models

When we talk on preservation of natural environments, we usually have to take into account at least (1) the pollution connected with the human activity, and specifically with production, (2) the exploitation of natural resources, which may be connected with pollution, but also with deteriorating its natural beauty and scenic wilderness, and (3) that part of the capital, which is connected with the above activities. Quality of the environment is also connected with (4) population growth which leads to congestion. In this chapter we concentrate on reviewing some basic models describing these four activities and notions connected with them. For other reviews the reader is referred to [67] and [8].

To allow for better comparison, all values below are expressed in monetary equivalents.

2.1 A Prototype Basic Model

We start with the following prototype optimization model

$$maximize\{\int_0^\infty e^{-\delta t}U(C,Q,P)dt\} \tag{2.1}$$

constrained by

$$\dot{P}(t) = H(K,L,Q,P,A) \tag{2.2}$$

$$\dot{R}(t) = G(R,K,L,Q) \tag{2.3}$$

$$\dot{K}(t) = W(K,C,A,Q,L,I) \tag{2.4}$$

$$\dot{L}(t) = S(L,C,P) \tag{2.5}$$

$$P(0) = P_0,\ R(0) = R_0,\ K(0) = K_0,\ L(0) = L_0 \tag{2.6}$$

where U is a so-called utility function, C is consumption, P is a pollution stock, R represents remaining reserves of a resource and Q its extraction, K is the capital, I is the investment, and L is the labour (as a constant part of population), all in time t. A are the expenses connected with the abatement of pollution. H, G, S, W are some, unspecified yet, functions. The discount rate δ says how much weight we give to the utility function in the future, as compared to the present value. Small δ ($e^{-\delta t}$ decreasing slowly) is connected with a higher evaluation of the utility function in the future. Big δ ($e^{-\delta t}$ decreasing quickly) is connected with neglection of the future values of the utility function (the myopic point of view).

Let us assume that

$$\frac{1}{L}H(K,L,Q,P,A) = h(k,q,p,a) \tag{2.7}$$

$$\frac{1}{L}G(R,K,L) = g(r,k) \tag{2.8}$$

$$\frac{1}{L}W(K,C,A,Q,L) = w(k,c,a,q) \tag{2.9}$$

$$\frac{1}{L}S(L,C,P) = s(c,p) \tag{2.10}$$

Then the above model can be transformed to per capita values

$$p(t) = \frac{P(t)}{L(t)}, r(t) = \frac{R(t)}{L(t)}, k(t) = \frac{K(t)}{L(t)} \tag{2.11}$$

$$c(t) = \frac{C(t)}{L(t)}, i(t) = \frac{I(t)}{L(t)}, a(t) = \frac{A(t)}{L(t)} \tag{2.12}$$

As generally, for $x(t) = X(t)/L(t)$

$$\dot{x} = \frac{\dot{X}}{L} - \frac{X}{L}\frac{\dot{L}}{L} \tag{2.13}$$

then the equations (2.2) - (2.4), (2.6) take the following form

$$\dot{p}(t) = -s(c,p)p(t) + h(k,c,q,a) \tag{2.14}$$

$$\dot{r}(t) = -s(c,p)r(t) + g(r,k) \tag{2.15}$$

$$\dot{k}(t) = -s(c,p)k(t) + w(k,c,a,q) \tag{2.16}$$

$$p(0) = p_0,\ r(0) = r_0,\ k(0) = k_0 \tag{2.17}$$

This transformation is used if growth of labour has to be considered, and is particularly useful when $s(c,p) = s$ is a constant (the geometrical growth of labour). For the constant labour the equations (2.2), (2.3) and (2.4) are rather used.

Let us notice that the assumptions (2.7)–(2.10) can be satisfied for some classes of functions. Take for example functions that are linear in their arguments. Other classes will be considered in the sequel.

In the next sections we shall discuss the elements of the above model in more details.

2.2 Capital

The following mathematical model of the capital accumulation is due to Ramsey [58]. It has the form

$$\dot{K}(t) = -\alpha K(t) + I(t) \tag{2.18}$$

where α is the rate of the capital depreciation. As before, K is the capital and I is the investment. Let us assume that the output of the economy is given by a production function

$$Y = e^{\theta t}F(K,L,R) \tag{2.19}$$

where θ is a rate of technical progress. The production is divided as follows

$$Y(t) = C(t) + I(t) + A(t) \tag{2.20}$$

Inserting $I(t)$ from above in (2.18) we get

$$\dot{K}(t) = -\alpha K(t) + e^{\theta t}F(K,L,R) - C(t) - A(t) \tag{2.21}$$

An often used form of the production function is *the CES (constant elasticity of substitution) production function*

$$F(K,L,R) = [\beta_1 K^{(\sigma-1)/\sigma} + \beta_2 L^{(\sigma-1)/\sigma} + \beta_3 R^{(\sigma-1)/\sigma}]^{\sigma/(\sigma-1)} \tag{2.22}$$

where $\beta_i \geq 0$, $i = 1,2,3$, $\beta_1 + \beta_2 + \beta_3 = 1$ and $0 < \sigma < \infty$, $\sigma \neq 1$.

A special case of the above, for $\sigma \to 1$, is of importance. We have

$$F(K,L,R) = e^{\sigma/(\sigma-1)\ln[\beta_1 K^{(\sigma-1)/\sigma} + \beta_2 L^{(\sigma-1)/\sigma} + \beta_3 R^{(\sigma-1)/\sigma}]} \tag{2.23}$$

Denote the exponent above by $\zeta(\sigma)$. Using the de L'Hospital rule we have

$$\lim_{\sigma\to 1} \zeta(\sigma) = \lim_{\sigma\to 1} \frac{\ln[\beta_1 K^{(\sigma-1)/\sigma} + \beta_2 L^{(\sigma-1)/\sigma} + \beta_3 R^{(\sigma-1)/\sigma}]}{(\sigma-1)/\sigma} =$$

$$= \lim_{\sigma\to 1} \frac{1/\sigma^2[\beta_1 K^{(\sigma-1)/\sigma}\ln K + \beta_2 L^{(\sigma-1)/\sigma}\ln L + \beta_3 R^{(\sigma-1)/\sigma}\ln R]}{1/\sigma^2[\beta_1 K^{(\sigma-1)/\sigma} + \beta_2 L^{(\sigma-1)/\sigma} + \beta_3 R^{(\sigma-1)/\sigma}]} =$$

$$= \beta_1 \ln K + \beta_2 \ln L + \beta_3 \ln R \tag{2.24}$$

Then

$$\lim_{\sigma\to 1} F(K,L,R) = e^{\beta_1 \ln K + \beta_2 \ln L + \beta_3 \ln R} = K^{\beta_1} L^{\beta_2} R^{\beta_3} \tag{2.25}$$

This limit function is called *the Cobb-Douglas production function*

$$F(K,L,R) = K^{\beta_1} L^{\beta_2} R^{\beta_3} \tag{2.26}$$

As $\lim_{\sigma\to\infty} \frac{\sigma-1}{\sigma} = 1$, then

$$\lim_{\sigma\to\infty} F(K,L,R) = \beta_1 K + \beta_2 L + \beta_3 R \tag{2.27}$$

Also transforming this in a similar way as in the previous paragraph we get for $\sigma \to 0$

$$\lim_{\sigma\to 0} F(K,L,R) = \min\{K,L,R\} \tag{2.28}$$

Thus taking into account the above limits we can define the CES function for all values of $0 \leq \sigma \leq \infty$.

Let us notice that for all the above production functions the transformations of the kind (2.7)–(2.10) are valid.

The production function in the model (2.21) must be specified before its solution is attempted. The solution is induced by the factors present in the production function and therefore the model has an exogenous character. Recently some interest emerged in specifying an endogenous capital model. It started with the works of Romer [60], Lucas [41], Barro [2] and Rebelo [59], see also [3], and is often referred to as *Rebelo* or *Barro-Rebelo model.* It is assumed in it that the production function is proportional to the capital, that is

$$Y(t) = aK(t)$$

This way the model (2.21) takes the form

$$\dot{K}(t) = (a-\alpha)K(t) - C(t) - A(t) \tag{2.29}$$

which in many instances will be much easier to cope with than the model (2.21). Notice that formally this model can be also obtained from (2.22) taking $\beta_1 = 1$ or from (2.28) under assumption on the limiting role of the capital on the production function.

2.3 Resources

The problem of natural resources exhaustion was a subject of interest already in the 19th century when Malthus discussed the food limitation for the growth of human population, with exponentially growing population and limited agricultural land. The mathematical treatment of this subject began, however, with the work of Hotelling [32] in 1931. In 1950s Gordon [27] and Schaefer [61] discussed some economic questions connected with fishery, treated as a common resource. The Hotelling ideas were also continued by Gordon [28]. In 1970s, partially due to the Club of Rome activity (see [48], [49]) and the oil crises, the problem of resources exhaustion became quite a popular subject in the economic literature. Many now classical papers were presented at the *Symposium on the Economics of Exhaustible Resources* and printed in *The Review of Economic Study*, [15], [68], [70], [71]. This subject is also extensively discussed in books, see e.g. [9], [17], [29], [65], [21], [12]. The detailed model depends very much on the particular case considered, see e.g. [55]. However, a rough classification of cases is possible. This is what will be done in the sequel.

2.3.1 Nonrenewable Resources

Nonrenewable resources are usually related to mining. The models of nonrenewable resources usually are classified into two groups: *pure depletion* or *exhaustible* resources, and nonrenewable resources with possibility of *discoveries*.

Exhaustible Resources

The model in this case takes the very simple form

$$\dot{R}(t) = -Q(t) \tag{2.30}$$

$$R(0) = R_0 \tag{2.31}$$

where $Q(t)$ is extraction of the resource. The extraction may depend on the demand D which can then be a function of the price $\pi(t)$, which leads to the dependence in the form

$$\dot{R}(t) = -D(\pi(t)) \tag{2.32}$$

Under *competition* a so called *Hotelling rule* is often applied

$$\frac{\dot{\pi}(t)}{\pi(t)} = \delta \tag{2.33}$$

which connects the change of price with the "inflation" coefficient δ. The Hotelling rule gives the simple equation for prices $\pi(t) = \pi(0)e^{\delta t}$. Then denoting by T the time when reserves will be exhausted we go to the following simple problem

$$\int_0^T Q(t)dt = R_0 \tag{2.34}$$

$$Q(T) = D(\pi(0)e^{\delta T}) = 0 \tag{2.35}$$

Now the latter equation may be solved for $\pi(0)$ which then allows us to calculate T from the former.

This solution is usually compared either with *the monopolistic one*, i.e. when the price is determined by a monopolist, who would wish to maximize

$$\int_0^T e^{-\delta t}[\pi(Q(t))Q(t) - Cs(Q(t))]dt \tag{2.36}$$

subject to (2.30), where Cs is the cost of extraction; or with *the social optimum one*, where the problem is to maximize

$$\int_0^T e^{-\delta t}U(Q(t))dt \tag{2.37}$$

subject to (2.30), where U is a suitably defined utility function. Both the above are optimal control problems which can be further analyzed using e.g. maximum principle conditions and phase plane analysis as given in the Appendices, see also the literature cited above.

Exploration and Discoveries

The problem can be slightly more complicated if *exploration* and *discoveries* may augment reserves. Let then $X(t)$ represent cumulative discoveries. We may now write a model for discoveries

$$\dot{X}(t) = d(E(t), X(t)), \quad X(0) = X_0 \tag{2.38}$$

where d is a discovery-rate function depending on $E(t)$ – the exploratory effect, which usually will be a function of the capital allocated for exploration. Including the effect of discoveries in (2.30) we get

$$\dot{R}(t) = d(E(t), X(t)) - Q(t), \quad R(0) = R_0 \tag{2.39}$$

This case then leads to a two-state ((2.38) and (2.39)) optimization problem. Note that the objective function should now include the cost of exploration.

Research and Development

Another possibility to enhance the time of using an exhaustible resource is in change of technology. This is usually connected with research and development in the area. Constant progress can be described by the rate θ connected with the production function (2.19). Here we consider the case of one change in technology. Assume that the breakthrough occurs at time T. This will influence consumption C and extraction of the resource Q, and possibly also the utility function U. Then the objective function takes the form

$$\int_0^T e^{-\delta t}U_1(C_1, Q_1)dt + \int_T^\infty e^{-\delta t}U_2(C_2, Q_2)dt \tag{2.40}$$

Time T depends on the level of knowledge $X(t)$. We assume that the level necessary for the breakthrough is X_s, that is the time T is given by

$$T : X(T) = X_s \tag{2.41}$$

Accumulation of knowledge may be described by a differential equation dependent on the capital $K_D(t)$ allocated to the research and development

$$\dot{X}(t) = D(K_D(t)), \quad X(0) = 0 \tag{2.42}$$

If this equation is a deterministic one, then T can be found from the relation

$$\int_0^T D(K_D(t))dt = X_s \tag{2.43}$$

and the problem reduces to an optimal control problems with a slightly more complicated objective function (2.40).

However, we may consider that actually the dependence (2.42) is a stochastic one, and therefore T is a stochastic variable, see [15], [18], [34]. We assume then that it is desired to optimize the expected value of the objective function (2.40)

$$E\{\int_0^T e^{-\delta t} U_1(C_1, Q_1)dt + \int_T^\infty e^{-\delta t} U_2(C_2, Q_2)dt\} \tag{2.44}$$

with respect to the stochastic variable T. Moreover, we assume that the probability of discovery of the new technology is known and described by the probability density function $\omega(T)$, such that

$$\int_0^\infty \omega(T)dT = p_D \leq 1 \tag{2.45}$$

This means that we allow for the nonnegative probability $p_N = 1 - p_D$ that the new technology will not be discovered at all. Now we can write the objective function as

$$p_N \int_0^\infty e^{-\delta t} U_1(C_1, Q_1)dt + \int_0^\infty \omega(T)[\int_0^T e^{-\delta t} U_1(C_1, Q_1)dt + \int_T^\infty e^{-\delta t} U_2(C_2, Q_2)dt]dT \tag{2.46}$$

Denoting

$$\int_T^\infty e^{-\delta t} U_2(C_2, Q_2)dt = W(C_2(T), Q_2(T)) \tag{2.47}$$

and integrating the second (nested) integral in (2.46) by parts we get

$$p_N \int_0^\infty e^{-\delta t} U_1(C_1, Q_1)dt + \int_0^\infty e^{-\delta t} U_1(C_1, Q_1)dt \int_0^\infty \omega(t)dt - \int_0^\infty e^{-\delta t} U_1(C_1, Q_1) \int_0^t \omega(\tau)d\tau dt +$$
$$+ \int_0^\infty \omega(T) W(C_2(T), Q_2(T))dT = \int_0^\infty e^{-\delta t} U(C_1, Q_1)[p_N + \Omega(t)]dt + \int_0^\infty \omega(t) W(C_2, Q_2)dt \tag{2.48}$$

where $\Omega(t) = \int_t^\infty \omega(\tau)d\tau$. This way the objective function (2.44) is reduced to the deterministic one, without averaging. This again allows us to apply the optimal control theory approach. Let us, however, observe that the requirements of knowledge of the probability characteristics, necessary for determining p_N and $\omega(\tau)$, may be limiting its use.

2.3.2 Renewable Resources

Renewable resources consist of biological populations which can reproduce and grow, or some inanimate resources which are subject to supplementary flux, like water or wind.

Growth Function

The models connected with this kind of resources need to take into account the nature of growth of the resource. This may require rather thorough knowledge of the underlying phenomena, like biology of the species considered or climate and geology of the terrain. As usual in the economic literature, we consider that the amount of the resource stock $R(t)$ changes, when unaltered, according to a differential equation

$$\dot{R}(t) = G(R(t)), \quad R(0) = R_0 \tag{2.49}$$

where G is called *the growth function*. In the case of biological populations examples of such models may, for example, comprise:

- the Malthus [45] model $\dot{R}(t) = \gamma R(t)$, with a constant γ,
- the Verhulst [74] model $\dot{R}(t) = \gamma R(t)[1 - \frac{R(t)}{W}]$, with constants γ, W,
- the Gompertz [26] model $\dot{R}(t) = \gamma R(t) \ln(\frac{W}{R(t)})$, with constants γ, W,
- the Monod [51], [52] model $\dot{R}(t) = \gamma R(t) \frac{S(t)}{W+S(t)}$, with constants γ, W, where $S(t)$ is the amount of substance which limits the growth (like e.g. food),
- the Lotka-Volterra [40], [75] predator-prey model $\dot{R}_1(t) = \gamma R_1(t) - \alpha R_1(t) R_2(t)$, $\dot{R}_2(t) = \beta R_1(t) R_2(t) - \kappa R_2(t)$,, with constants $\gamma, \alpha, \beta, \kappa$ (in this case two species are taken into account),

and many other, see also Section 2.5 Labour.

In the case of nonbiological resource often stochasticity of the supplementary flux has to be considered. This will not be considered here.

Harvesting

The renewable resource may be harvested. This notion, typically used for biological populations, can also mean exploitation of an inanimate resource, e. g. ground water, with supplementary flow. The rate of harvest (yield) Y per unit time will usually depend on "effort" $E(t)$ and the available stock, such that

$$Y(t) = M(E(t), R(t)) \tag{2.50}$$

Then the relation (2.49) takes the form

$$\dot{R}(t) = G(R(t)) - Y(t) = G(R(t)) - M(E(t), R(t)), \quad R(0) = R_0 \tag{2.51}$$

A possible concept here is to keep the resource on a constant level. Then $\dot{R}(t) = 0$ and we get the set of algebraic equations

$$G(R) - Y = 0, \quad Y = M(E, R) \tag{2.52}$$

Which, after having eliminated R, allow us to find the sustained yield function $Y = Y(E)$ and the level of the resource stock $R(E)$. With the above, given the stock R, the yield can be optimized.

More generally we can introduce an objective function (usually a benefit) to find an optimal effort. Here also the monopolist and social optimum can be considered. An important issue, however, is to avoid overexploitation of the resource, which may result in *stock extinction*. This in particular may easily happen when no regulation is imposed (*common-property resource*). When connected with the capital equation a question of *optimal investment strategy* may arise, see [12].

2.4 Pollution

Pollution models are often even more difficult to build than those for resources. First of all, the harmful influence of a pollutant is usually related to its concentration, which may be difficult to calculate knowing the emissions. Moreover, the emission of some pollutants may be difficult to estimate, as it may be distributed on bigger regions or, even more troublesome, related with some processes of not fully known nature. Secondly, the flows of the pollutant may be of a rather complicated nature, including dispersion in air, surface and/or underground water, or soil, with possibility of interactions. It can travel with the receiving media to remote sites and influence environment far from the place of their disposal. The site of the pollutant's deposit may depend on random phenomena, like meteorological conditions. Some pollutants can also accumulate in living organisms, giving rise to its propagation in food chains. Finally, the pollutants are usually subject to different chemical and/or biological transformations, often only little known. Fundamentals of modelling and many examples of models can be found in [33].

The early papers on economic analysis of dynamic models of pollutants and waste accumulations appeared at the beginning of 1970s, [36], [57], [66], [14], [44], [62], [76]. Probably the best exposition of the early models is in [25], see also the review paper [67].

The early papers on economic dynamic pollution problems took a rather simplistic models of pure accumulation of the pollutant or accumulation with a possibility of its *degradation*

$$\dot{P}(t) = E(t) - \beta P(t), \quad P(0) = P_0 \tag{2.53}$$

where P is the pollutant stock, $\beta \geq 0$ is the rate of its degradation and E is emission. The emission is often connected with the production, as it is frequently understood that the pollutants are mostly emitted as wastes in the production processes.

Unlike in the resource sector, here *abatement* of the pollution is possible (which may also include *recycling* of the waste). This is mainly connected with capital K_A allocated to the abatement technology. Then the model takes the form

$$\dot{P}(t) = E(t) - \beta P(t) - A(K_A), \quad P(0) = P_0 \tag{2.54}$$

Important issues in the pollution modelling are connected with the technical progress and international dimension of polluting activity.

2.5 Labour

When longer times and bigger territories, like countries, continents or the globe, are considered, the change in labour due to growth of population has to be considered. The models here are usually of the exogeneous kind - they do not depend on other model variables. For deeper description of population models see [37].

The most popular model used in the economic considerations is *the Malthus model* [45] of exponential growth. It has the form

$$\dot{L}(t) = \gamma L(t), \quad L(0) = L_0 \tag{2.55}$$

where γ is a constant, called *the intrinsic rate of growth.* It has a simple solution, very suitable for calculation of per capita values

$$L(t) = L_0 e^{\gamma t} \tag{2.56}$$

This model describes the constant growth of the population and therefore has only limited applications, mainly for shorter time horizons. There are other models which take into account some saturation effects. For example *the Verhulst model* [74]

$$\dot{L}(t) = \gamma L(t)[1 - \frac{L(t)}{W}], \quad L(0) = L_0 \tag{2.57}$$

where γ and W are constants, and W is called *the carrying capacity of the environment.* Its solution is a famous logistic curve

$$L(t) = \frac{W}{1 + \frac{W-L_0}{L_0} e^{-\gamma t}} \tag{2.58}$$

with the limit $\lim_{t\to\infty} L(t) = W$. Another famous model of this kind is *the Gompertz model* [26]

$$\dot{L}(t) = \gamma L(t) \ln(\frac{W}{L(t)}), \quad L(0) = L_0 \tag{2.59}$$

where γ and W are constants. Its solution takes the form

$$L(t) = W(\frac{L_0}{W})^{e^{-\gamma t}} \tag{2.60}$$

and again we have $\lim_{t\to\infty} L(t) = W$.

More complicated models take into account the age distribution in the population. The models, mainly developed in the first half of this century, include integral, matrix, difference and partial differential equations ones. This subject is extensively studied in demography, see e.g. [37], [38].

2.6 Utility Function

The shape of the utility function is important in the theoretical analysis of the economic problems in environment, as visualized in Appendix B. The important characteristics for the analysis are the second (partial) derivatives U_{XX} of the utility function with respect to any of its argument X. In the stochastic setting the following notions are often used:

- *risk aversion*, if $U_{XX} < 0$ (U concave in X),
- *risk preference*, if $U_{XX} > 0$ (U convex in X),
- *risk neutrality*, if $U_{XX} = 0$ (U linear in X).

Another important characteristic may be *separability* of U, e.g. when $U(C,P) = U_1(C) + U_2(P)$.

Detailed definition of the utility function is basically related to the possibility of measuring the benefits and costs of environmental policy. While it may be comparatively easier to do for many resource problems involving goods which can be bought and sold on markets, and which thus can be easily expressed in monetary terms, this is hardly the case in pollution problems. The direct estimation is sometimes possible when the cost of abatement is considered (like expenditures on cleaner fuels, abatement control equipment, etc.) or after damages caused by pollution have occurred (usually in forestry, fishing, and agriculture). But even then the possibility of adjustment of firms or individuals to the changes caused by pollution has to be considered. Such adjustment can, for example, include irrigation of the land or alterations in the cultivated plants or number of acres planted, as well as passing on part of the cost increase to consumers.

The direct estimation is, however, not possible for such goods as clean air or water. To find values of this kind of goods some indirect methods of measurements have been developed. The detailed discussion of these methods is out of the scope of this paper. It can be found, for instance, in a recent survey [13], from where also the classification given below has been taken.

Let us introduce the damage function $S(P,Z)$ that links pollution P and another factor Z with some, perhaps abstract, value S. S may be, for example, time spent on recreation, swimming or boating on a lake, or the number of average days in a year with some respiratory problems due to air pollution. Z may be connected with some actions undertaken to mitigate the effect of pollution, like willingness to travel to a lake being more far than a close, but polluted one. Or moving to another part of the city, or another city. Or just buying some medicine to relieve the respiratory symptoms due to air pollution. Now, with a known change in pollution it is possible to measure the corresponding changes of the factor Z. Thus, if Z is expressed in monetary equivalent, it allows us to calculate willingness of people to pay for improvement in environmental standard. This approach was called *the averting behaviour approach.*

A closely related way, called *the weak complementarity approach*, is connected with valuating factors which are complementary to environment quality, like for example more visits to a recreational site when the source of pollution was removed and the environment cleaned, or increase in the household prices there.

A third approach use *hedonic market methods.* This approach exploits the concept of hedonic prices. This is a notion that the price of a good can be decomposed into the prices of the attributes that make up the good. For example, the price of a house may include air quality, or wages for a job may include risk of death. Then regressing the prices on the corresponding attributes the value of the clear environment can be estimated.

Although sometimes imprecise, this valuation techniques has been applied in many practical problems, see [13], and may help in developing the utility function for a specific case.

2.7 Environment Regulation

A central idea in pollution control is connected with the fact that the firms, looking for optimal production conditions in a competitive environment, may discharge excessive wastes, engaging this way in excessive polluting activities. Let us consider an example which for simplicity is a static case, that is the variables are constant in time. We assume that all the functions below are sufficiently smooth. Assume now that a firm discharges a waste with emission E. The waste contains a pollutant such that its flow to the environment is $P(E)$. The firm production function F depends on some input K (which may be the capital, the labour etc.), the emission E and pollution P, i.e. its form is $F(K,E,P(E))$. We adopt some conditions

$$\frac{\partial F}{\partial E} \geq 0, \quad \frac{\partial F}{\partial P} \leq 0, \quad \frac{\partial P}{\partial E} > 0 \tag{2.61}$$

Moreover we assume that the functions above are concave. The optimal emission for the firm can be found from the first order necessary optimality conditions

$$\frac{dF}{dE} = \frac{\partial F}{\partial E} + \frac{\partial F}{\partial P}\frac{dP}{dE} = 0 \tag{2.62}$$

The social optimum will involve some utility function $U(F,P)$ and we assume that

$$\frac{\partial U}{\partial F} > 0, \quad \frac{\partial U}{\partial P} < 0 \tag{2.63}$$

and that the function U is concave. Now, the first order necessary optimality conditions are

$$\frac{dU}{dE} = \frac{\partial U}{\partial F}\left(\frac{\partial F}{\partial E} + \frac{\partial F}{\partial P}\frac{dP}{dE}\right) + \frac{\partial U}{\partial P}\frac{dP}{dE} = 0 \tag{2.64}$$

or

$$\frac{\partial F}{\partial E} + \frac{\partial F}{\partial P}\frac{dP}{dE} + \frac{\partial U}{\partial P}\frac{dP}{dE}/\frac{\partial U}{\partial F} = 0 \tag{2.65}$$

Let us denote the social optimum by E^S. Due to the assumptions we have

$$\frac{dF}{dE}|_{E^S} = \left(\frac{\partial F}{\partial E} + \frac{\partial F}{\partial P}\frac{dP}{dE}\right)|_{E^S} = -\left(\frac{\partial U}{\partial P}\frac{dP}{dE}/\frac{\partial U}{\partial F}\right)|_{E^S} > 0 \tag{2.66}$$

We see that the social optimum does not give the optimal solution for the firm. Moreover, to get the optimal solution it would be necessary for the firm to increase the emission of wastes. Thus the firm optimizing its production discharges excessive waste. To reach the social optimum some additional restrictions on emission should be imposed.

The problem of externalities, like the one in the above example, was considered by Pigou [56] who proposed to use taxes as a regulator. In the above example, denoting the tax for the excessive emission by λ, we get the new function for the firm to be optimized

$$F(K,E,P(E)) + \lambda(E - E^S) \tag{2.67}$$

with the optimality condition

$$\frac{dF}{dE} = \frac{\partial F}{\partial E} + \frac{\partial F}{\partial P}\frac{dP}{dE} + \lambda = 0 \tag{2.68}$$

This condition is equal to the social optimum condition if

$$\lambda = \frac{\partial U}{\partial P}\frac{dP}{dE}/\frac{\partial U}{\partial F} \tag{2.69}$$

As more parties are involved the problem becomes more difficult and, moreover, complications arise connected with imperfect information and defensive activities of the victims. That is why the Pigouvian taxes have been repeatedly attacked. The early criticism is due to Coase [10], whose argument was that the distortions associated with the externalities would be resolved through bargaining among interested parties. This may be, however, difficult to implement in many conflicts involving pollution. Some other actions, like subsidies for lowering the emission, marketable emission permits, effluent charge etc., were discussed. There is also a possibility to use legal liabilities after pollution effects has been found. A good source of discussion of different actions and their impacts are [64] and [4]. For review of problems related with the policy instruments and reaction of the involved actors as well as of the literature see also [13].

As the direct approach involving calculation of damages caused by some parties to other constitute big obstacles, other solutions seem more practically relevant. A popular approach is to determine first some standards for environmental quality which enables then to design a regulatory system to achieve these standards. This often leads to so called command-and-control policy, as opposed to economic incentives mentioned above.

Chapter 3

A One State Variable Pollution Problem

Although theoretically possible, the detailed analysis of the problem (2.1) – (2.6) seems rather cumbersome. Various simpler models have been therefore proposed and analyzed. The simplest one with the one state variable will be reviewed in this chapter.

If the labour growth is constant or given exogeneously (and then eliminated), then there are three possible variations of the problem (2.1) – (2.6) giving rise to one state variable problems. These are: with the pollution accumulation model, with the capital accumulation model, and with the model describing dynamics of a resource extraction. We discuss here only the first of them. The early formulations of this problem is due to Plourde [57] and Keeler at al. [36] but its main analysis was done by Forster [25]. Besides the discussion of the standard equilibrium model we give here also full discussion of the nonstandard models with no equilibrium points but only steady state stationary points on the boundaries. The discussion of other models can be found elsewhere. The book [21] is a good source where also discussion of the Ramsey model of capital accumulation can be found, together with pollution treated as a static variable. Various resource extraction models are also extensively discussed in [9] and [12].

3.1 Assumptions and Problem Formulation

The utility function is here expressed as a smooth (twice continuously differentiable, including continuities at the boundaries of the feasibility set) finite value function which depends on only two variables: consumption and pollution, i.e. it has the form $U(C, P)$. It is also assumed that in the interior and on the boundary (except possibly in few places where it will be clearly stated) of the set of feasible solutions

$$U_C > 0, \;\; U_{CC} < 0, \;\; U_P < 0, \;\; U_{PP} < 0, \;\; U_{CP} \leq 0 \tag{3.1}$$

For some technical proofs we assume sometimes that the higher cross derivatives of the above function are zero, i. e.

$$U_{C^k P^l} = 0 \quad \text{for} \quad k > 0, l > 0, k + l > 2 \tag{3.2}$$

Notice that this assumption is always true when the function U is separated in C and P. We call the functions satisfying the above condition *weakly connected in arguments*. It is further assumed that a fixed output $K < \infty$ is produced over the time. It is allocated to consumption C and pollution control (abatement) K_A so that

$$K = C(t) + K_A(t) \tag{3.3}$$

The stock of pollution increases according to the equation (2.54) discussed in the section 2.4, which in this case has the form

$$\dot{P}(t) = E(C(t)) - A(K_A(t)) - \beta P(t), \;\; P(0) = P_0 > 0 \tag{3.4}$$

A function Z is again defined as a smooth (as in the case of U) finite function

$$Z(C) = E(C) - A(K - C) \tag{3.5}$$

and the following assumptions are taken in the interior and on the boundary (with some exceptions as described before) of the set of feasible controls

$$Z'(C) > 0, \;\; Z''(C) > 0 \tag{3.6}$$

and we assume that Z is bounded on $[0, K]$. We also assume that there exists C_0 such that $Z(C_0) = 0$ and that $0 < C_0 < K$. Then

$$Z(C) = \begin{cases} < 0 & \text{if} \quad 0 \le C < C_0 \quad \text{(net abatement)} \\ = 0 & \text{if} \quad C = C_0 \quad \text{(net sustainability)} \\ > 0 & \text{if} \quad C_0 < C \le K \quad \text{(net pollution)} \end{cases} \tag{3.7}$$

Thus the problem can be formulated as follows

$$\max_C \int_0^\infty e^{-\delta t} U(C, P) dt, \;\; \delta > 0 \tag{3.8}$$

$$\dot{P} = Z(C) - \beta P, \;\; P(0) = P_0 > 0 \tag{3.9}$$

$$C \ge 0, \;\; K - C \ge 0, \;\; P \ge 0 \tag{3.10}$$

Notice that because of the assumptions taken

$$\int_0^\infty e^{-\delta t} U(C, P) dt \le \int_0^\infty e^{-\delta t} U(K, 0) dt \le U(K, 0)/\delta < \infty \tag{3.11}$$

Thus the integral exists.

Different simpler formulations of the above problem has been considered, for instance as illustrative examples in [12] or [63], often with some economic interpretation. The simplifications mainly include separation in parameters of the utility function and often linearity of the equation of motion (3.9). An extension of this problem, for the case when the utility function depends on the derivative (rate) of the pollution accumulation $\dot{P}$ was discussed in [73]. However, in this paper the utility function is separable and the equation of motion is linear.

Let us now define the (current value) Hamiltonian and Lagrangean functions (see Appendix A)

$$\tilde{H}(C, P, \lambda) = \lambda_0 U(C, P) + \lambda[Z(C) - \beta P] \tag{3.12}$$

$$L(C, P, \lambda, \mu_1, \mu_2, \mu_3) = \tilde{H}(C, P, \lambda) + \mu_1 C + \mu_2 (K - C) + \mu_3 P \tag{3.13}$$

Then the necessary (maximum principle) optimality conditions are

$$C^* = \arg\max_C \tilde{H}(C, P^*, \lambda^*) \tag{3.14}$$

$$L_C = \lambda_0 U_C + \lambda^* Z'(C^*) + \mu_1 - \mu_2 = 0 \tag{3.15}$$

$$\dot{\lambda}^* = (\delta + \beta)\lambda^* - U_P - \mu_3 \tag{3.16}$$

$$\mu_1 C^* = \mu_2 (K - C^*) = \mu_3 P^* = 0, \;\; \mu_i \ge 0, \;\; i = 1, 2, 3 \tag{3.17}$$

$$(\lambda_0, \lambda^*) \ne (0, 0), \quad \lambda_0 = 0 \quad \text{or} \quad 1 \tag{3.18}$$

and additionally (3.9) – (3.10).

3.2 Some Solution Properties

3.2.1 Discussion of λ_0

Let us consider first the case $\lambda_0 = 0$. In this case the solution does not depend on the utility function but is determined by constraints. Indeed, the Hamiltonian (3.12) now is

$$\tilde{H}(C,P,\lambda) = \lambda[Z(C) - \beta P] \tag{3.19}$$

From (3.18) we have $\lambda^* \neq 0$. When $\lambda^* > 0$, then the maximum of $\tilde{H}(C,P^*,\lambda^*)$ is at the value maximizing $Z(C)$, that is at $C = K$ (full pollution). Solving the equation (3.9) we get in this case

$$P(t) = P_0 e^{-\beta t} + \frac{Z(K)}{\beta}(1 - e^{-\beta t} > 0 \tag{3.20}$$

Then from (3.17) $\mu_3 = 0$ and from (3.16) we get at the steady state

$$\lambda^* = \frac{U_P}{\delta + \beta} < 0 \tag{3.21}$$

As λ^* is continuous in t, this contradicts our assumption $\lambda^* > 0$.

Now assume $\lambda^* < 0$. Then the value maximizing the Hamiltonian is $C = 0$ (full abatement). In this case, as $Z(0) < 0$

$$P(t) = \begin{cases} P_0 e^{-\beta t} + \frac{Z(0)}{\beta}(1 - e^{-\beta t}) & \text{for} \quad t < \frac{1}{\beta}\ln(1 - \frac{\beta P_0}{Z(0)}) \\ 0 & \text{otherwise} \end{cases} \tag{3.22}$$

But then, for $t \geq \frac{1}{\beta}\ln(1 - \frac{\beta P_0}{Z(0)})$, the abatement $K - C_0$ is sufficient which means that the sustainable value $C = C_0 > 0$ can be shifted to consumption, increasing this way the utility function. Then $C = 0$ does not maximize the Hamiltonian. This contradiction finally eliminates the case $\lambda_0 = 0$ and therefore we have $\lambda_0 = 1$.

3.2.2 Solution at $P = 0$

We start this case again with discussion of solutions on the boundaries. Solution of the equation (3.9) is

$$P(t) = P_0 e^{-\beta t} + \int_0^t e^{-\beta(t-\tau)} Z[C(\tau)]d\tau \tag{3.23}$$

Then, if $C \geq C_0$ (i.e. $Z > 0$), then $P(t) > 0$ for all finite t. If not, then there exist at least some interval at the beginning where $P(t) > 0$ (because $P_0 > 0$ and Z is bounded). Therefore $\mu_3 = 0$, at least for a sufficiently small t. Let us also notice as a byproduct of this analysis that as by assumption $Z(C) < Z(K)$, then $P(t) < P_0 e^{-\beta t} + \frac{1}{\beta}Z(K)(1 - e^{-\beta t})$. Then solution of the above differential equation is bounded and the asymptotical solution is bounded by $Z(K)/\beta$.

Let us, however, notice that the steady state (equilibrium) solution may or may not be on a boundary $P = 0$, dependent on some function characteristics [24]. For this let us consider the costate-state (λ - P) plane (see Appendix B) and assume that $\mu_i = 0, i = 1,2,3$. For the curve $\dot{\lambda}^* = 0$ we have from (3.16) and (3.1)

$$\lambda^* = \frac{U_P}{\delta + \beta} < 0 \quad \text{for} \quad P > 0 \tag{3.24}$$

Now, differentiating (3.15) with respect to P we get

$$C'(P) = -\frac{U_{CP}}{U_{CC} + \lambda^* Z''(C)} \geq 0 \tag{3.25}$$

and, finally, differentiating (3.24) with respect to P yields

$$\lambda'^*(P) = \frac{U_{PP} + U_{CP}C'(P)}{\delta + \beta} < 0 \tag{3.26}$$

So the curve $\lambda^*(P)$ is strictly decreasing. Its value for $P \to 0^+$ is given by

$$\lim_{P\to 0^+} \lambda^*(P) = \frac{\lim_{P\to 0^+} U_P(C,P)}{\delta + \beta} = \lambda^\lambda \tag{3.27}$$

Let us now consider the curve $\dot{P} = 0$. From (3.9) we have

$$P = \frac{Z(C)}{\beta} \tag{3.28}$$

and then

$$P'(\lambda^*) = \frac{Z'(C)C'(\lambda^*)}{\beta} \tag{3.29}$$

Again from (3.15), differentiating now with respect to λ^*

$$U_{CC}C'(\lambda^*) + U_{CP}P'(\lambda^*) + Z'(C) + \lambda^* Z''(C)C'(\lambda^*) = 0 \tag{3.30}$$

which, after some small algebraic manipulations, gives

$$P'(\lambda^*) = -\frac{[Z'(C)]^2}{\beta U_{CC} + \beta\lambda^* Z''(C) + U_{CP}Z'(C)} > 0 \tag{3.31}$$

So the curve $P(\lambda^*)$ is strictly increasing.

We can find the value of λ_P^* giving $P(\lambda_P^*) = 0$. At $P = 0$ we get $Z(C) = 0$ which is satisfied for $C = C_0$ (net sustainability assumption). Then from (3.15) we have

$$\lambda^P = -\frac{U_C(C_0, 0)}{Z'(C_0)} \tag{3.32}$$

Now (see also Fig. 3.1), if $\lambda^P < \lambda^\lambda$, then the curves for $\dot{\lambda}^* = 0$ and $\dot{P} = 0$ cut for $P > 0$ and the steady state value of pollution is positive. Otherwise the steady state value of pollution is 0 and the asymptotic stationary point[1] is $(\lambda_P^*, 0)$, with $\dot{\lambda}^* = 0$ for $\mu_3 = (\delta + \beta)\lambda^P - U_P(C_0, 0)$. A condition to assure the positive steady state pollution may be $\lambda^\lambda = 0$ which is for example satisfied for

$$\lim_{P\to 0^+} U_P(C,P) = 0 \qquad \text{for any } C \tag{3.33}$$

[1] Notice that the stationary point in [24] is misplaced due to not considering the constraint $P > 0$ there. There may be an asymptotic solution on a boundary, not violating the constraint, but this may happen only when the curves for $\dot{\lambda}^* = 0$ and $\dot{P}^* = 0$ cut at the boundary.

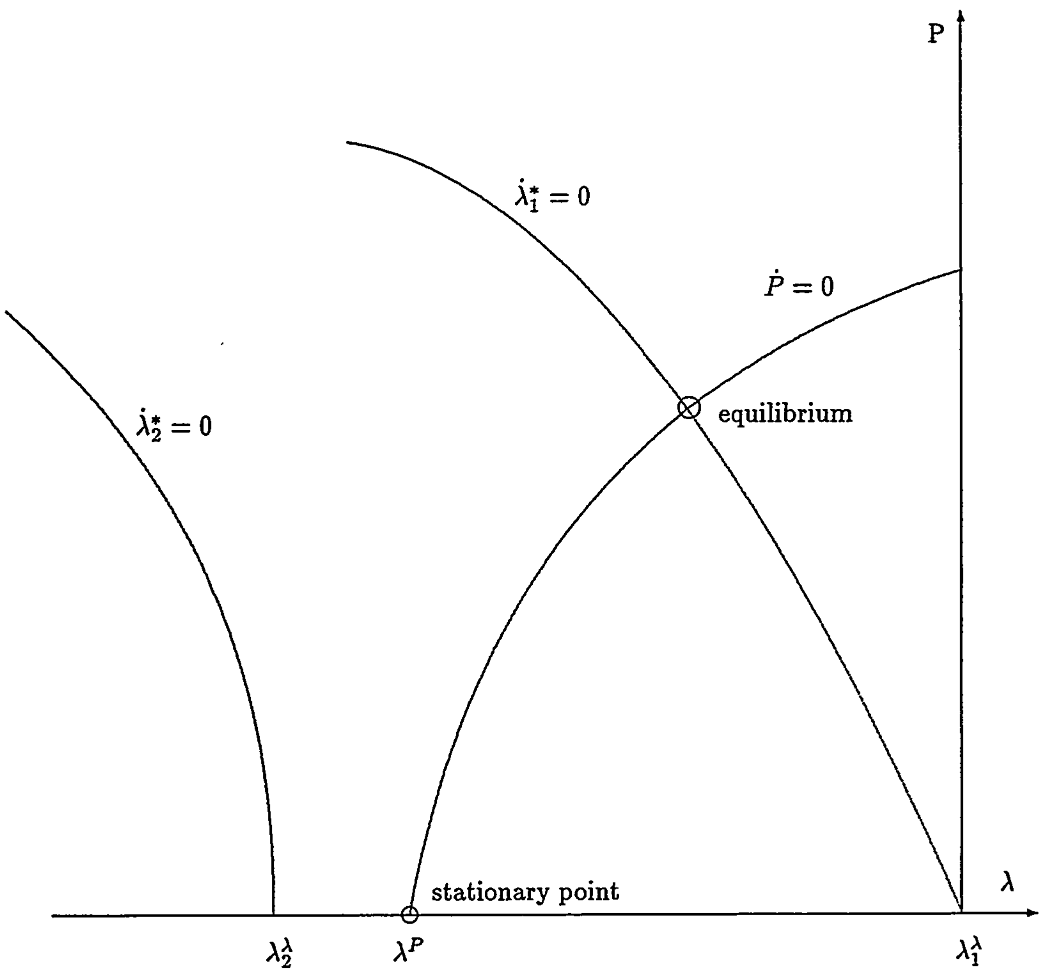

Figure 3.1: The equilibrium (case 1) and the stationary point (case 2) at the costate-state phase plane.

3.2.3 Solutions at $C = 0$ or $C = K$

From the equation (3.15) we get

$$\lambda^* = -\frac{U_C + \mu_1 - \mu_2}{Z'(C^*)} \tag{3.34}$$

Thus, if C^* is in the interior of the interval $[0, K]$ (where $\mu_1 = \mu_2 = 0$), then $\lambda^* < 0$ and the Hamiltonian is concave there. Then from the maximization in (3.14) we can get a solution in the interior or possibly at a boundary point.

Solutions at the boundaries 0 or K can be eliminated by making appropriate assumptions on the problem functions. As we have

$$Z'(C) = E'(C) + A'(K - C) \tag{3.35}$$

then

$$\lim_{K_A \to 0+} A'(C) = +\infty \;\;\Rightarrow\;\; \lim_{C \to K-} Z'(C) = +\infty \tag{3.36}$$

Now, U_C must be bounded at K because the function U is increasing and concave with respect to C. If we confine ourselves to finite μ_2 (or finite λ^*), then we get from (3.34) $\lambda^* = 0$. But this cannot be true all the time as from (3.16) we have in this case (notice that $\mu_3 = 0$)

$$\dot{\lambda}^* = -U_P > 0 \tag{3.37}$$

Thus with the assumption (3.36) the solution cannot be at $C = K$ for finite λ^*.

However, we cannot take a similar assumption at $C = 0$, because Z is increasing and convex. Therefore $Z'(0)$ must be finite. Instead, another assumption may be taken

$$\lim_{C \to 0+} U_C(C, P) = \infty \;\text{ for any }\; P > 0 \tag{3.38}$$

With the above from (3.15) either λ^* or μ_2 must be infinite. So it is impossible to satisfy the condition (3.15) for finite values of the involved variables.

3.2.4 Summary of Boundary Solutions

Summarizing, the following assumptions are sufficient to keep the solution out of the boundaries (for finite values of variables):

$$\lim_{P \to 0+} U_P(C, P) = 0 \qquad \text{for any } C \text{ (to have } \; P > 0) \tag{3.39}$$

$$\lim_{C \to K-} Z'(C) = +\infty \qquad \text{(to have } \; C < K) \tag{3.40}$$

$$\lim_{C \to 0+} U_C(C, P) = \infty \;\text{ for any } P > 0 \text{ (to have } \; C > 0) \tag{3.41}$$

A possible shape of the $Z(C)$ curve is depicted at Fig. 3.2.

Let us, finally, check the constraint qualification. Calculating the matrix (A.21) we get

$$\begin{bmatrix} 1 & C & 0 & 0 \\ -1 & 0 & K - C & 0 \\ P_C & 0 & 0 & P \end{bmatrix} \tag{3.42}$$

The troubles with the full rank may occur when $P_C = 0$ for $P = 0$ and either $C = 0$ or $C = K$. From (3.15) we have for $P = 0$

$$\dot{P}_C = Z'(C) \tag{3.43}$$

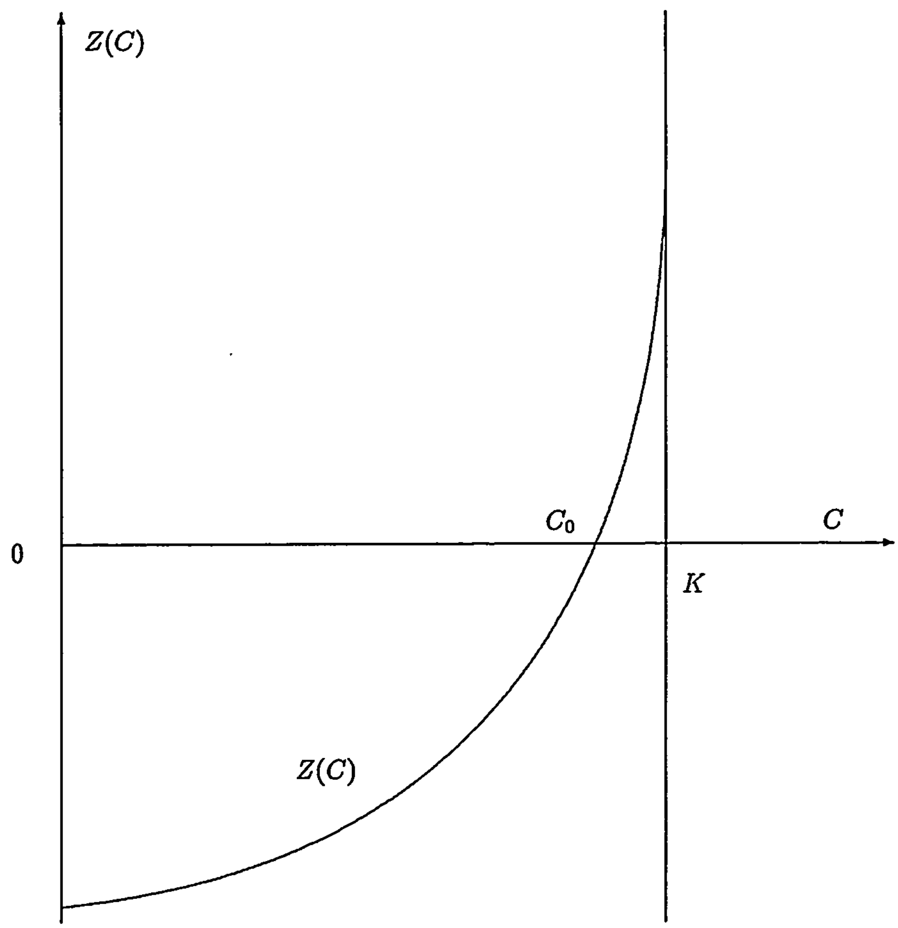

Figure 3.2: A possible shape of the function $Z(C)$.

Then $\dot{P}_C > 0$ except possibly at $C = 0$, if $Z'(0) = 0$. The latter case is, however, not feasible with our assumptions on $Z(C)$ because then $P = 0$ and $C = 0$ (full abatement under no pollution). Thus, even when $P_C = 0$ at some t', then it could only be at this specific time, with $P_C > 0$ for $t > t'$. We can then conclude that the constraint qualifications are satisfied.

3.3 Phase Plane Analysis – Equilibrium at an Interior Point

Let us now analyze the solution on the phase planes. This actually will follow closely the steps presented in the Appendix B (notice, however, that now the assumptions on the equation of motion function are different). We assume first that the equilibrium is in the interior of the feasible set, that is $\mu_i = 0, i = 1, 2, 3$. The equilibrium is given by the stationary point of the equations (1.8), (1.14) and (1.151)

$$Z(C^*) = \beta P^* \tag{3.44}$$

$$U_C + \lambda^* Z'(C^*) = 0 \tag{3.45}$$

$$(\delta + \beta)\lambda^* - U_P = 0 \tag{3.46}$$

Notice that the solution to the above equations exists and is unique because both $Z'(C^*) > 0$ and $Z''(C^*) > 0$.

3.3.1 State-Costate Phase Plane

The equilibrium point at the costate-state phase plane has been analyzed at the Fig. 3.1. However, we change now the coordinates to comply with the Appendix B. The isoclines $\dot{\lambda}_1^* = 0$ and $\dot{P} = 0$ divide the orthant into four isosectors I, II, III, IV, see Fig. 3.3. Denote

$$Z(C(\lambda, P)) - \beta P = N(\lambda, P) \quad \{= \dot{P}\} \tag{3.47}$$

We have

$$N_P = Z'(C(\lambda, P))C_P - \beta \tag{3.48}$$

But, taking into account the partial derivative of (3.15) with respect to P we get

$$C_P = -\frac{U_{CP}}{U_{CC} + \lambda Z''(C(\lambda, P))} \leq 0 \tag{3.49}$$

so that

$$N_P < 0 \tag{3.50}$$

This means that to the right of the curve $\dot{P} = 0$ (in the isosectors III and IV) there is $\dot{P} < 0$ and to the left (in the isosectors I and II) $\dot{P} > 0$.

Similarly denote

$$(\delta + \beta)\lambda - U_P(C(\lambda, P), P) = M(\lambda, P) \quad \{= \dot{\lambda}\} \tag{3.51}$$

and calculate

$$M_\lambda = \delta + \beta - U_{CP} C_\lambda \tag{3.52}$$

But then the partial derivative of (3.15) with respect to λ is

$$C_\lambda = -\frac{Z'(C(\lambda, P))}{U_{CC} + \lambda Z''(C(\lambda, P))} > 0 \tag{3.53}$$

so that

$$M_\lambda > 0 \tag{3.54}$$

and $\dot{\lambda} > 0$ above the curve $\dot{\lambda} = 0$ (i.e. in the isosectors I and IV) and $\dot{\lambda} < 0$ below the curve $\dot{\lambda} = 0$ (i.e. in the isosectors II and III). The results of the above analysis is also presented on the Fig. 3.3. We see that there are two convergent orbits, o_2 and o_4 which are stable solutions to our equations. For a given P_0, choosing appropriately $\lambda(0)$ at $t = 0$ on the optimal path it is possible to enter one of these optimal paths which lead to to the equilibrium. The optimal consumption is calculated by maximizing the Hamiltonian.

3.3.2 State-Control Phase Plane

Let us now analyze the state - control (P - C) phase plane. The curve $c_1(P)$ for $\dot{P} = 0$ is given by

$$Z(c_1(P)) - \beta P = 0 \tag{3.55}$$

so we have

$$c_1'(P) = \frac{\beta}{Z'(C)} > 0 \tag{3.56}$$

Thus it is increasing.

To find the equation for $\dot{C}$ let us differentiate (3.15) with respect to t which yields

$$U_{CC}\dot{C} + U_{CP}\dot{P} + \dot{\lambda}Z'(C) + \lambda Z''(C)\dot{C} = 0 \tag{3.57}$$

and insert for $\dot{\lambda}$ from (3.16) and for λ again from (3.15) to get

$$\dot{C} = \frac{(\delta+\beta)U_C + U_P Z'(C) - U_{CP}\dot{P}}{U_{CC} - U_C Z''(C)/Z'(C)} \tag{3.58}$$

Thus the curve $c_2(P)$ for $\dot{C} = 0$ has to satisfy the following equation (notice that U_C, U_P and U_{CP} depend on $c_2(P)$ as well)

$$(\delta+\beta)U_C + U_P Z'(c_2(P)) - U_{CP}\dot{P} = 0 \tag{3.59}$$

Now, taking the partial derivative with respect to P we have

$$\begin{aligned}(\delta+\beta)[U_{CC}c_2'(P) + U_{CP}] + [U_{CP}c_2'(P) + U_{PP}]Z'(c_2)+\\ +U_P Z''(c_2)c_2'(P) - [U_{C^2P}c_2'(P) + U_{CP^2}]\dot{P} - U_{CP}\dot{P}_P = 0\end{aligned} \tag{3.60}$$

From (3.9) we get

$$\dot{P}_P = Z'(C)c_2'(P) - \beta \tag{3.61}$$

so at the equilibrium (i.e. for $\dot{P} = 0$), or for U being weakly connected in the arguments we finally have

$$c_2'(P) = -\frac{\delta U_{CP} + U_{PP}Z'(C)}{(\delta+\beta)U_{CC} + U_P Z''(C)} < 0 \tag{3.62}$$

Thus the equilibrium point is (at least locally - but we soon learn that it is globally) unique.

Moreover, denoting (we use the same letters as previously but obviously the functions are different)

$$M(C,P) = Z(C) - \beta P \tag{3.63}$$

$$N(C,P) = (\delta+\beta)U_C + U_P Z'(C) - U_{CP}\dot{P} \tag{3.64}$$

we get

$$M_P = -\beta < 0 \tag{3.65}$$

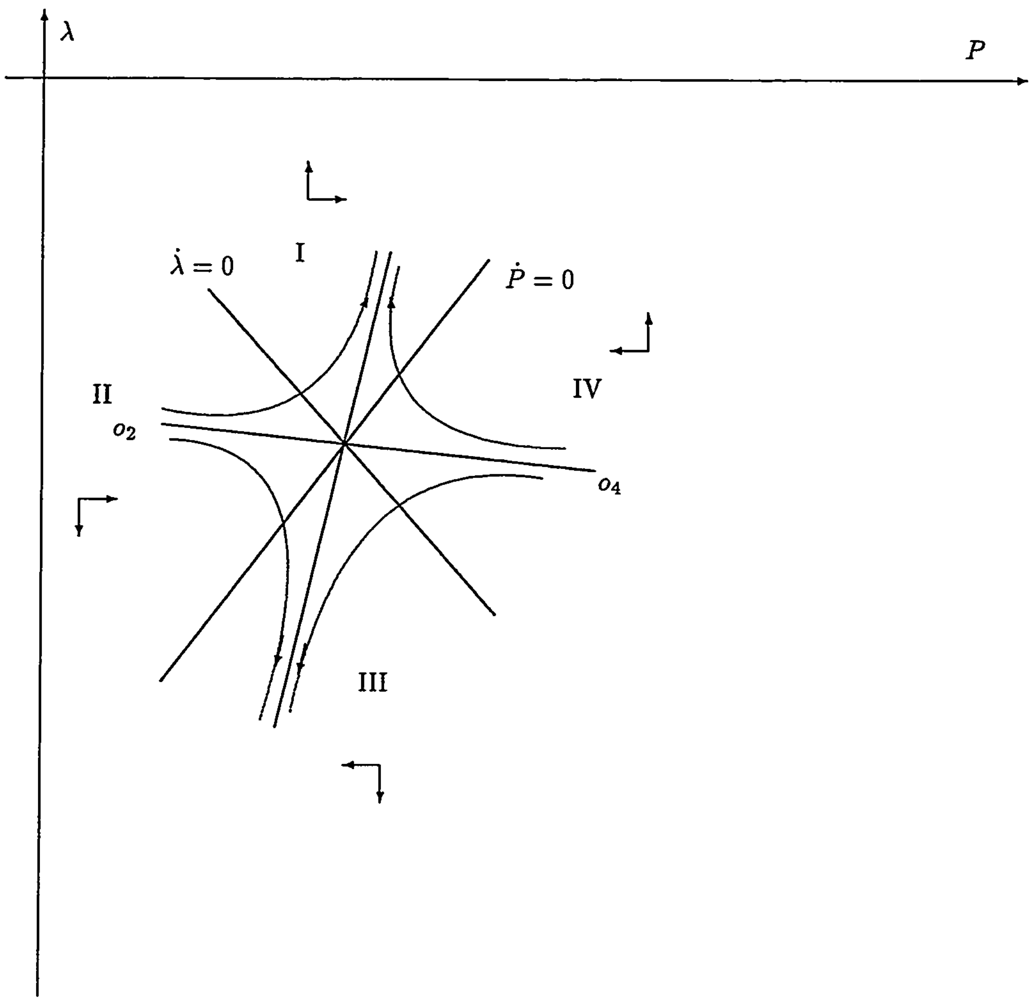

Figure 3.3: Analysis of the equations on the state-costate phase plane.

so $\dot{P} < 0$ in the isosectors III and IV, and $\dot{P} > 0$ in the isosectors I and II. Similarly, at $\dot{P} = 0$ (or for U being weakly connected in the arguments)

$$N_C = (\delta + \beta)U_{CC} + U_P Z''(C) < 0 \tag{3.66}$$

As the denominator in (3.58) is negative and does not change its sign in the vicinity of the equilibrium, then $\dot{C} > 0$ in the isosectors I and IV and $\dot{C} < 0$ in the isosectors III and IV (see Fig. 3.4).

Again, the optimal path depends on P_0. For $P_0 > P_\infty$ the optimal solution may lie, for smaller t, partly on the axis (i.e. for $C = 0$ – full abatement case). This will not be the case when (3.41) is satisfied. Also for $P_0 < P_\infty$ the solution may be partly on the boundary for smaller t (for $C = K$). This will not be the case when (3.40) is true. Notice, however, that both (3.40) and (3.41) are only the sufficient conditions, that is the solution may not be on the respective boundaries even when they are not satisfied. A possible optimal trajectory lying partially on the boundaries is depicted at the Fig. 3.4.

Let us notice that from (3.58) we may characterize the equilibrium in the following way. It exists if the solution to the equation

$$(\delta + \beta)U_C(C^*, \frac{Z(C^*)}{\beta}) + U_P(C^*, \frac{Z(C^*)}{\beta})Z'(C^*) = 0 \tag{3.67}$$

satisfies $C_0 \le C^* \le K$. Then

$$P^* = \frac{Z(C^*)}{\beta}, \quad \lambda^* = -\frac{U_C(C^*, \frac{Z(C^*)}{\beta})}{Z'(C^*)} \tag{3.68}$$

Let us rewrite the equation (3.67) in the following way and denote the left side by $L(C^*)$ and the right side by $R(C^*)$

$$L(C^*) = \frac{U_C(C^*, \frac{Z(C^*)}{\beta})}{Z'(C^*)} = -U_P(C^*, \frac{Z(C^*)}{\delta + \beta} = R(C^*) \tag{3.69}$$

We have

$$L'(C^*) = \frac{[U_{CC} + U_{CP}\frac{Z'(C^*)}{\beta}]Z'(C^*) - U_C Z''(C^*)}{[Z'(C^*)]^2} < 0 \tag{3.70}$$

and

$$R'(C^*) = -\frac{U_{CP} + U_{PP}\frac{Z'(C^*)}{\beta}}{\delta + \beta} > 0 \tag{3.71}$$

Thus the left side is decreasing in C^* while the rigtht side is increasing.Therefore they may cut at most in one point and there may be not more than one equilibrium.

Notice, however, that the height and slope of the $R(C^*)$ depend on the discount factor δ, as presented on Fig. 3.5. We see that raising δ (move towards the myopic point of view) increases the solution value of consumption C^*. Reduction of δ (higher concern for the future) moves the optimal consumption closer to the zero pollution value.

As there is at most one equilibrium point, one might be interested what happens to the paths diverging away from the equilibrium. We know that the asymptotic solution for P is bounded and then the set of possible asymptotic solutions on the state - control phase plane is also bounded. So the paths cannot diverge to infinity here. The evolution of these paths will be discussed when the boundary stationary points are considered.

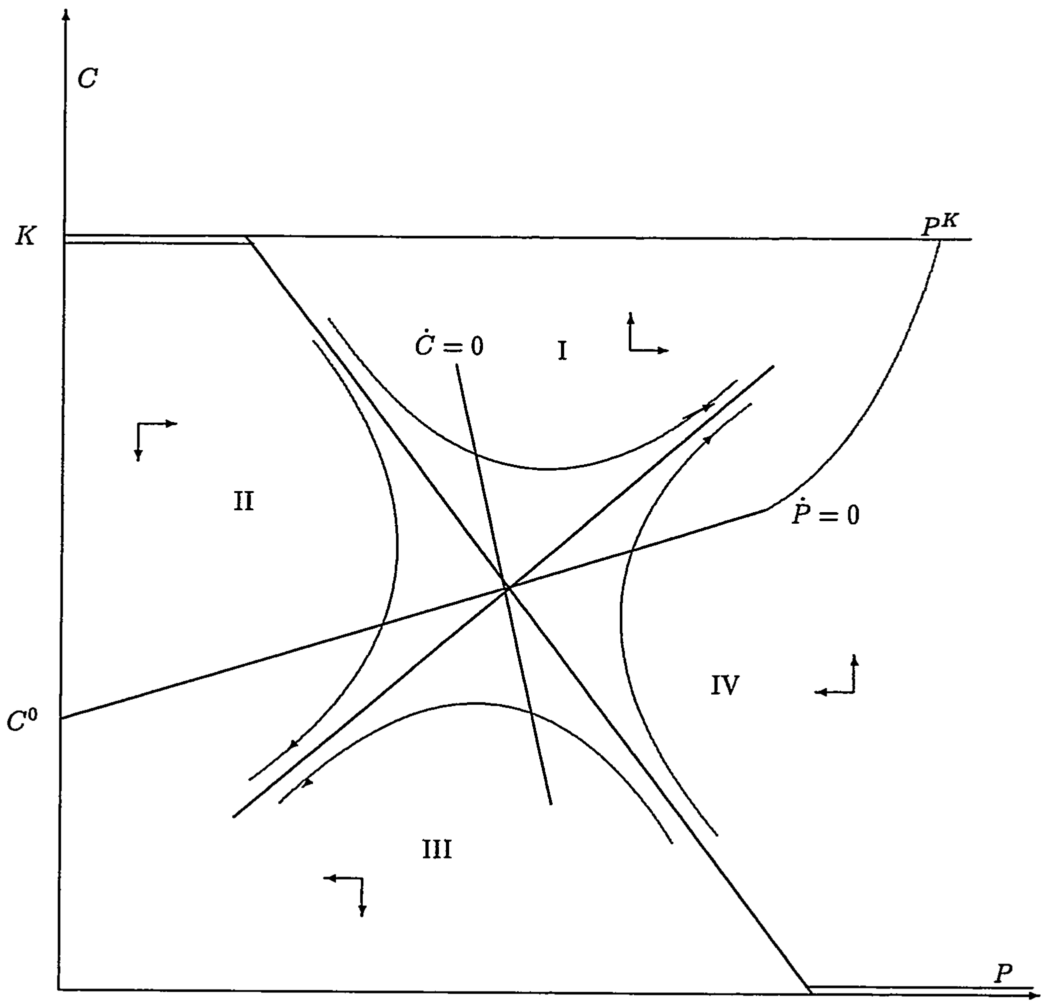

Figure 3.4: Analysis of the equations on the state-control phase plane.

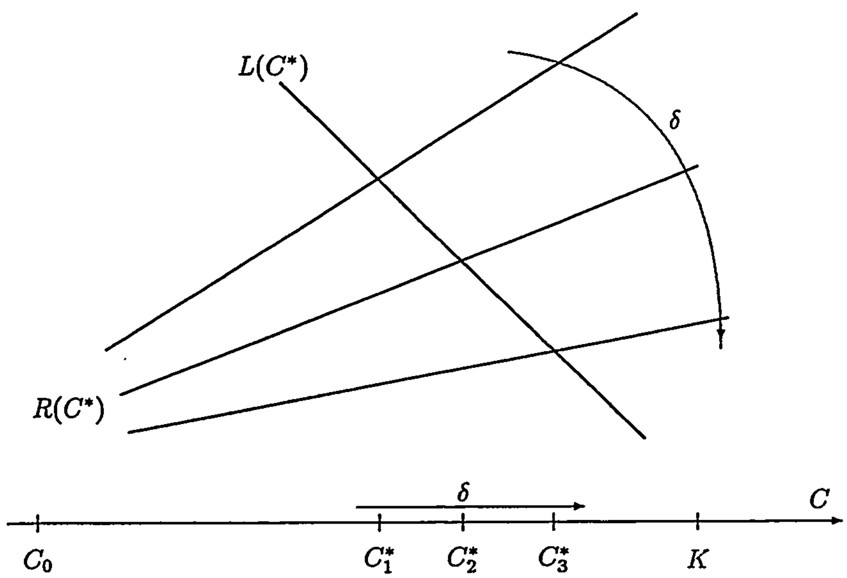

Figure 3.5: Dependence of the optimal consumption on the value of δ.

3.4 Phase Plane Analysis – Stationary Solution on Boundaries

3.4.1 Discussion of Stationary Solutions on Boundaries

Let us consider now cases when the asymptotic solution is on a boundary. The analysis of the cases when the equilibrium is on a boundary (the point is attained asymptotically) does not differ significantly from the one for the equilibrium in the interior. However, if a boundary is attained at the finite time, then the asymptotic analysis of the Appendix B is not valid, as there is no equilibrium. Yet we can still follow some analysis of possible asymptotic solutions on the phase plane.

A finite steady state stationary solution at boundaries must satisfy the following equations

$$Z(C^*) - \beta P^* = 0 \tag{3.72}$$

$$U_C + \lambda^* Z'(C^*) + \mu_1 - \mu_2 = 0 \tag{3.73}$$

$$(\delta + \beta)\lambda^* - U_P - \mu_3 = 0 \tag{3.74}$$

Let us check all boundaries in turn.

For $P = 0$ we have from (3.72) $Z(C) = 0$, that is $C = C_0$ which is on neither of boundaries $C = 0$ or $C = K$, then $\mu_1 = \mu_2 = 0$. Moreover we have there

$$\lambda^P = -\frac{U_C(C_0, 0)}{Z'(C_0)} < 0, \quad \mu_3 = (\delta + \beta)\lambda^P - U_P(C_0, 0) \tag{3.75}$$

So there may be a stationary point[2] $(C_0, 0, \lambda^P)$ on the boundary $P = 0$ if the following inequality holds

$$(\delta + \beta)U_C(C_0, 0) + U_P(C_0, 0)Z'(C_0) < 0 \tag{3.76}$$

Let us interpret this inequality geometrically on the state - control phase plane, see Fig. 3.6. From (3.58) on the line $P = 0$ we have

$$[U_{CC}(C, 0) - U_C(C, 0)Z''(C)/Z'(C)]\dot{C} = (\delta + \beta)U_C(C, 0) + U_P(C, 0)Z'(C) - U_{CP}(C, 0)\dot{P} \tag{3.77}$$

Then, if the inequality in (3.76) becomes equality, then the point $(C_0, 0)$ is en equilibrium because $\dot{P} = 0$ there, and therefore also $\dot{C} = 0$ must hold. Now, for $C = C_0$ we have $(\delta + \beta)U_{CC}(C_0, 0) + U_P(C_0, 0)Z''(C_0) < 0$ (this is also valid on the whole line if U is weakly connected in the arguments) so that the right hand side decreases with increase of C. Therefore, if (3.76) is satisfied, then the only possibility to get $\dot{C} = 0$ is to move downwards. Then the curve $\dot{C} = 0$ cuts the line $P = 0$ below C_0. Similarly, when the inequality in (3.76) is reversed, then the curve $\dot{C} = 0$ cuts the line $P = 0$ above C_0, in which case there is an equilibrium in the interior of the feasible space. So there may be no stationary point, a stationary point or an equilibrium in $(C_0, 0)$, depending on problem functions.

On the state-costate phase plane from (3.76) we get

$$\lambda^\lambda = \frac{U_P(C_0, 0)}{\delta + \beta} < -\frac{U_C(C_0, 0)}{Z'(C_0)} = \lambda^P \tag{3.78}$$

which means that the curve $\dot{\lambda} = 0$ cuts the axis $P = 0$ below the point where the curve $\dot{P} = 0$ cuts it.

[2] In [25] another point is claimed to be a stationary one for $P = 0$. However, it does not lie on the curve $\dot{P} = 0$.

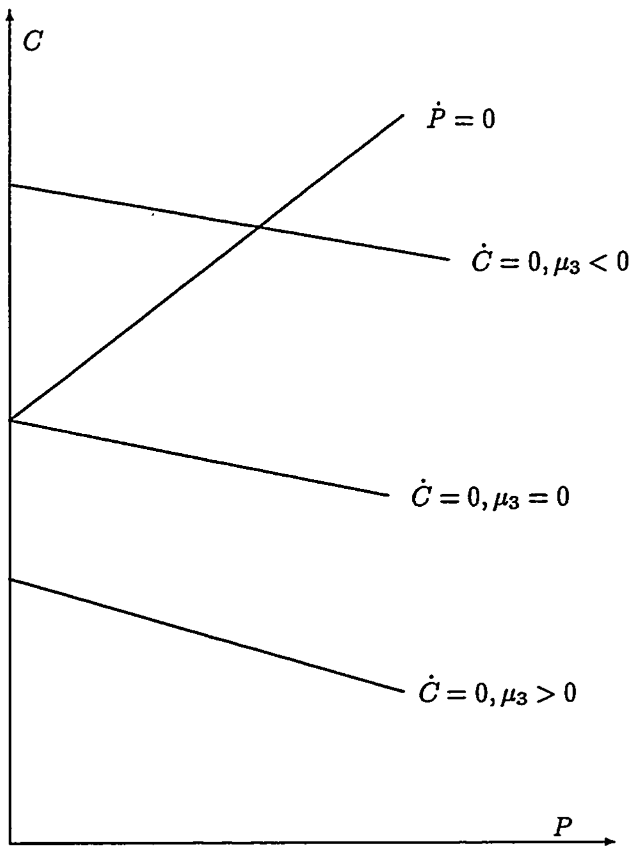

Figure 3.6: Dependence of the place of the curve $\dot{C} = 0$ in relation to the curve $\dot{P} = 0$ on the value of μ_3.

For $C = 0$ it is impossible to satisfy the equation (3.72) for $P \geq 0$. So there is no asymptotical stationary solution there.

For $C = K$ we have $\mu_1 = 0$, then from (3.72)

$$P^K = \frac{Z(K)}{\beta} \;\Rightarrow\; \mu_3 = 0 \tag{3.79}$$

from (3.74)

$$\lambda^S = \frac{U_P(K, P^K)}{\delta + \beta} < 0 \tag{3.80}$$

and from (3.73)

$$\mu_2 = U_C(K, P^K) + \lambda^S Z'(K) \tag{3.81}$$

So there is an asymptotic stationary solution here, if the following inequality holds

$$(\delta + \beta) U_C(K, \frac{Z(K)}{\beta}) + U_P(K, \frac{Z(K)}{\beta}) Z'(K) > 0 \tag{3.82}$$

To interpret geometrically the above inequality on the state-control phase plane, see Fig. 3.7, we again rewrite (3.58) for the line $C = K$ as

$$[U_{CC}(K, P) - U_C(K, P) Z''(K)/Z'(K)]\dot{C} = (\delta + \beta) U_C(K, P) + U_P(K, P) Z'(K) - U_{CP}\dot{P} \tag{3.83}$$

At the point (K, P^K) (or on the whole line for the function U weakly connected in arguments) the expression on the right hand side decreases with increasing P (notice that $\dot{P}_P = -\beta < 0$ there). If (3.82) holds, then to get $\dot{C} = 0$ it is necessary to move right (increase P). Thus the curve $\dot{C} = 0$ cuts the line $C = K$ at $P > P^K$. If the inequality in (3.82) is reversed, the curve $\dot{C} = 0$ cuts the line $C = K$ at $P < P^K$, when there is an equilibrium in the interior of the feasible space. If there is the equality, then the curve cuts the line at P^K and there is an equilibrium at the boundary, and it is attained asymptotically. We conclude that the point (K, P^K, λ^S) may be no stationary point, a stationary point[3] or the equilibrium.

On the state-costate phase plane from (3.82) we get

$$\lambda^K = -\frac{U_C(K, P^K)}{Z'(K)} < \frac{U_P(K, P^K)}{\delta + \beta} = \lambda^S < 0 \tag{3.84}$$

Which means that there is a stationary point there if the curve $\dot{\lambda} = 0$ cuts the line $P = P^K$ above the point where it is first reached by the curve $\dot{P} = 0$.

From the discussion of the condition (3.67) we conclude that there may be either the stationary point at $(C_0, 0)$ or at (K, P^K) and never in both of them at the same time.

Analyzing the dependence of the conditions (3.76), (3.78), (3.82) and (3.84) on the discount factor δ we see that increasing δ (moving towards the myopic point of view) we decrease the possibility of having the asymptotic stationary solution at the zero pollution consumption level C_0 while increasing the possibility of having it at no abatement level $C = K$.

3.4.2 State-Control Phase Plane

The Stationary Solution at $P = 0$

We consider first the case when the stationary point is at $(C_0, 0)$. The analysis of the slope of the curve $\dot{P} = 0$ done before is valid here, so the slope is positive. The curve cuts the axis C at the

[3] This stationary point was overlooked in [25].

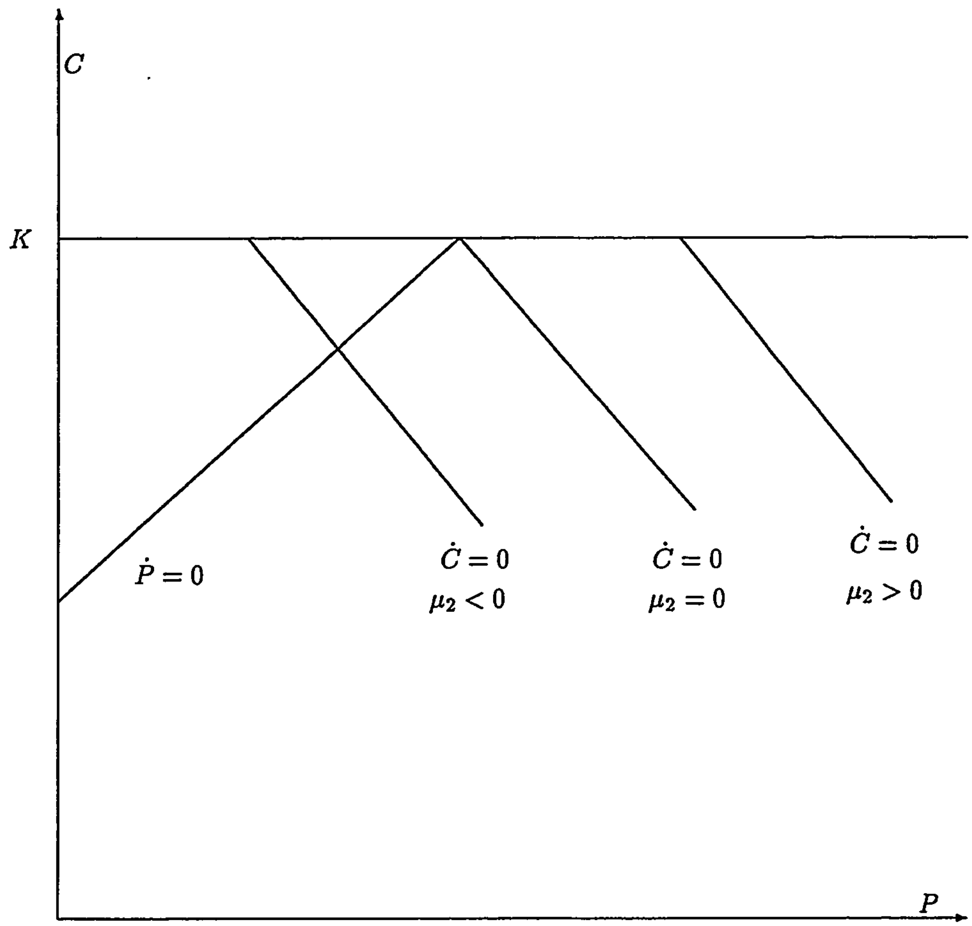

Figure 3.7: Dependence of the place of the curve $\dot{C} = 0$ in relation to the curve $\dot{P} = 0$ on the value of μ_2.

point C_0. It also cuts the horizontal line $C = K$ at the point P^K. The curve $\dot{C} = 0$ cuts the axis C at the point C^C. Its slope was calculated before only in the equilibrium and therefore the result obtained there is not valid here unless we assume that the function U is weakly connected in the arguments. We assume here that the slope is decreasing.

Now the curves $\dot{P} = 0$ and $\dot{C} = 0$ divide the set of feasible points (for $0 \leq C \leq K$, $P \geq 0$) into three isosectors denoted I, II, and III, see Fig. 3.8. The previous analysis of the signs of $\dot{P}$ is still valid here, so $\dot{P} > 0$ in the isosector I and $\dot{P} < 0$ in the isosectors II and III. To find the signs of $\dot{C}$ we should now determine the sign of the following expression

$$N_C = (\delta + \beta)U_{CC} + U_P Z''(C) - U_{C^2P}\dot{P} \tag{3.85}$$

which depends on the unknown sign of U_{C^2P}. We assume that N_C is positive, which is true at least when C^C is close to C_0 and P is close to 0 or the function U is weakly connected in the arguments. Then $\dot{\lambda} > 0$ at the isosectors I and II, and $\dot{\lambda} < 0$ at the isosector IV.

We see that the paths being close to the origin $(0,0)$ approach the axis C either directly, or first touching the axis P. Then they "jump" to the point $(C_0, 0)$. Among them there is a path going directly to this point, denoted on the figure o. The path lying above o converge to the point (K, P_K). They may cut the curve $\dot{P} = 0$ and approach the point from the left or approach the point from the right. Some of them may do it reaching first the line $C = K$.

We show now that the path o is optimal. The paths above it diverge and cannot be optimal (this claim will be proved when considering the state - costate phase plane). Let us consider then paths lying below it, which all satisfy the necessary optimality conditions, and specifically those which approach the axis C directly, not through the axis P (the latter can be, however, treated in a similar way). The points on the path move to some time, say t^0, to the axis C and after that they stay in the point $(C_0, 0)$. The time t^0 can be found from the equation (3.9), by separation of variables and integration, as

$$t^0 = \int_{P_0}^{0} \frac{dP}{Z(C) - \beta P} \tag{3.86}$$

Then the objective function can be written as

$$J = \int_0^{t^0} e^{-\delta t} U(C,P)dt + \int_{t^0}^{\infty} e^{-\delta t} U(C_0, 0)dt =$$

$$= \int_0^{t^0} e^{-\delta t} U(C,P)dt + \frac{1}{\delta} e^{-\delta t^0} U(C_0, 0) \tag{3.87}$$

Let C^0 be the value of C where the path touches the axis C. Now

$$\frac{dJ}{dC} = e^{-\delta t^0}[U(C^0, 0) - U(C_0, 0)]\frac{dt^0}{dC} + \int_0^{t^0} e^{-\delta t}[U_C(C,P) + U_P(C,P)p'(C)]dt \tag{3.88}$$

where $P = p(C)$ is the equation describing the path curve, solved for P. These can be done, as we assumed that this path is strictly decreasing. We have $p'(C) < 0$, and therefore the integral above is positive. Further, we have $U(C^0, 0) - U(C_0, 0) < 0$ because $C^0 < C_0$ and U is strictly increasing in C. Now

$$\frac{dt^0}{dC} = -\int_{P_0}^{0} \frac{Z'(C) - \beta p'(C)}{[Z(C) - \beta P]^2} dP < 0 \tag{3.89}$$

Then we conclude that

$$\frac{dJ}{dC} > 0 \tag{3.90}$$

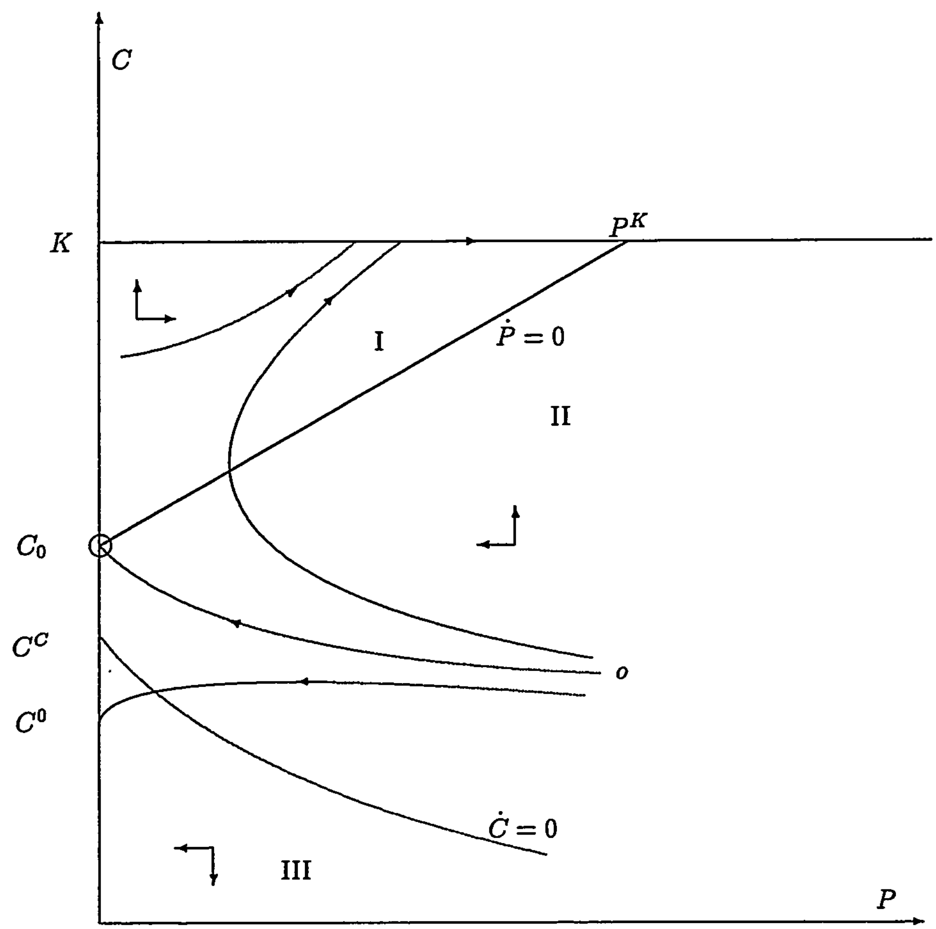

Figure 3.8: Analysis of the equations on the state-control phase plane for the case when the stationary solution is on the boundary $P = 0$.

so every path lying above gives better value of J. From these o is then the best.

Let us also notice that the paths which diverge from the point $(C_0, 0)$ are finally trapped in the point (K, P^K) which is still a point of attraction although it is not optimal (does not satisfy the necessary conditions).

The Stationary Solution at $C = K$

The curves $\dot{P} = 0$ and $\dot{C} = 0$ divide now the set of feasible solutions to three isosectors: I, II and III. From the analysis done for the equilibrium point, under possible suitable assumption on the slope of the curve $\dot{C} = 0$, we have that $\dot{P} < 0$ to the right of the curve $\dot{P} = 0$, i.e. in the isosectors II and III, and $\dot{P} > 0$ to the left of it, i.e. in the isosector I. Similarly $\dot{C} > 0$ above the curve $\dot{C} = 0$, i.e. in the isosector III, and $\dot{C} < 0$ below it, i.e. in the isosectors I and II. We see, Fig.3.9, that there are only two converging (optimal) paths, lying on the boundary $C = K$. All other paths diverge away from the stationary point and go to the axis $P = 0$ below the curve $\dot{P} = 0$. These points on the axis cannot satisfy the necessary optimality conditions, as will be obvious from the analysis of the state - costate phase plane.

3.4.3 State-Costate Phase Plane

We start the discussion of this case with examining the structure of the state-costate phase plane. We consider the points satisfying the necessary conditions. For a given P the values λ corresponding to the consumptions from inside the feasible set are given by the relation

$$\lambda(C, P) = -\frac{U_C(C, P)}{Z'(C)} \tag{3.91}$$

As we have

$$\lambda_C = -\frac{U_{CC} Z'(C) - U_C Z''(C)}{[Z"(C)]^2} > 0 \tag{3.92}$$

then the region on the state-costate phase plane corresponding to the set of consumptions from inside the feasible set is given by

$$-\frac{U_C(0, P)}{Z'(0)} < \lambda < -\frac{U_C(K, P)}{Z'(K)} \tag{3.93}$$

As

$$\lambda_P = -\frac{U_{CP}}{Z'(C)} \geq 0 \tag{3.94}$$

then both boundary curves are nondecreasing in P. Those values of λ which are above this region, i.e.

$$\lambda = \frac{-U_C(K, P) + \mu_2}{Z'(K)}, \quad \mu_2 \geq 0 \tag{3.95}$$

correspond to the points on the boundary $C = K$, and those below

$$\lambda = -\frac{U_C(0, P) + \mu_1}{Z'(0)}, \quad \mu_1 \geq 0 \tag{3.96}$$

correspond to the points on the boundary $C = 0$.

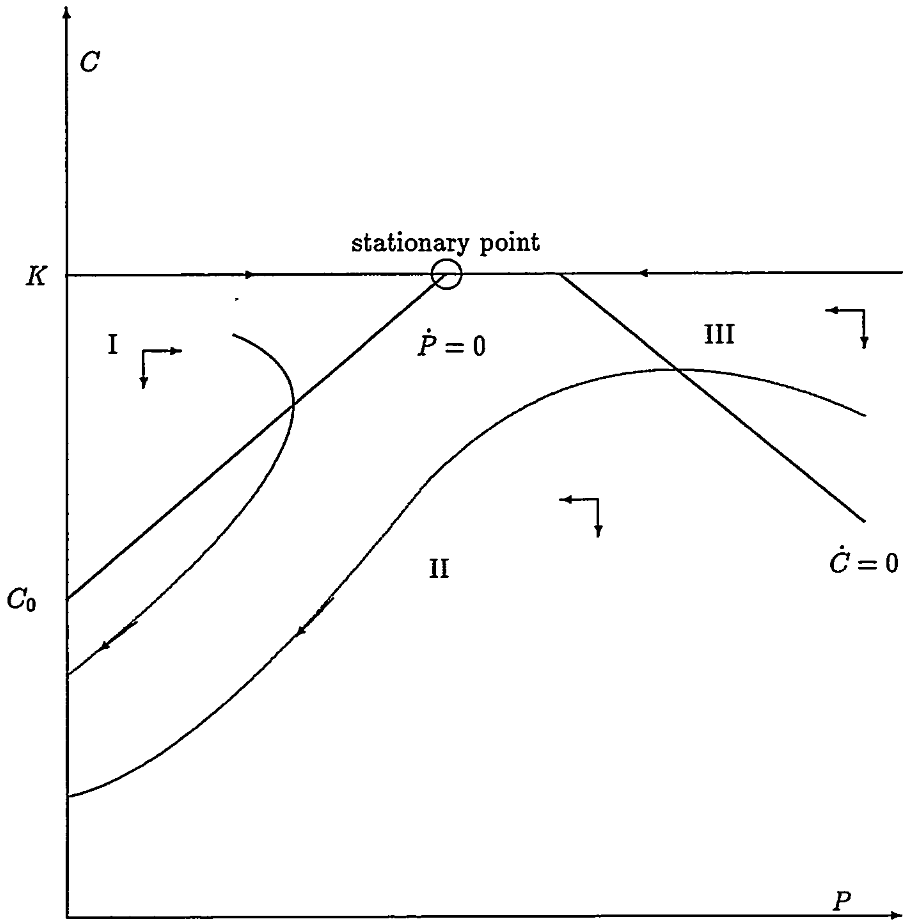

Figure 3.9: Analysis of the equations on the state-control phase plane for the case when the stationary point is on the boundary $C = K$.

The Stationary Solution at $P = 0$

The situation on the state - costate phase plane, for the case when the point at $(C_0, 0, \lambda^P)$ is the stationary one, is presented on the Fig.3.10. The curve $\dot{P} = 0$, after having reached the point (P^K, λ^K), which is on the boundary of the "feasible" region, goes further vertically, with the change of λ related only to the change of μ_2. As there is no asymptotic stationary point in this case at $C = K$, from (3.84) we find that $\lambda^S = \frac{U_P(K,P^K)}{\delta+\beta}$ is not higher than λ^K, that is in the "feasible" region. The curves $\dot{P} = 0$ and $\dot{\lambda}^* = 0$ divide the orthant into three isosectors denoted I, II and III. The previous analysis leading to (3.50) and (3.54) is still valid here, as long as $\lambda < -\frac{U_C(K,P)}{Z'(K)} \stackrel{\text{def}}{=} up(P)$, i.e. it is in the interior of the "feasible" region. Above $up(P)$ the value $C = K$ is constant and therefore its derivatives are equal to zero. Although both N_P and M_λ keep above $up(P)$ the same signs as below it, the derivatives on $up(P)$ may not (and as far as $U_{CP} < 0$ do not) exist. However, as both one-sided derivatives at any point of $up(P)$ (excluding, of course, the derivative N_P at P^K) exist and have the same sign, then those functions have there a generalized gradient (the subgradient for N and the supergradient for M), not containing 0. Although there may be then a jump in derivatives $\dot{P}$ and $\dot{\lambda}$ on $up(P)$ this cannot change the general direction of evolution of paths there. Therefore we have $\dot{P} < 0$ on the right side of the curve $\dot{P} = 0$ (isosectors II and III) and $\dot{P} > 0$ on the left side (isosector I). Also $\dot{\lambda}^* > 0$ above the curve $\dot{\lambda}^* = 0$ (isosectors I and II) and $\dot{\lambda} < 0$ below it (isosector III).

The paths entering the isosector I cannot stop at the P axis because we have there $\dot{\lambda}^* > 0$. So they cross it, which involves $C = K$ to fulfill (3.15), and continue with $C = K$ (otherwise (3.15) cannot be satisfied). As it cannot go back to negative values it must continue until $P^K = Z(K)/\beta$, which is a stationary solution of the evolution of the pollution stock equation (3.9). We show that these paths, as well as all other in the "boundary" region (for $C + K$) diverge with $\lambda^* \to \infty$ and that these paths are not optimal.

Let us consider the evolution of a path starting from a point $(P_0, \lambda(0))$ in the region where $C = K$, i.e. $\lambda(0) \geq up(P_0)$. Notice that because $\dot{\lambda} > 0$ there (under the understanding of derivatives on $up(P)$ as discussed above), then $\lambda(0) > -\frac{U_P(K,P_0)}{\delta+\beta}$. For this case we can solve the equation of motion as follows

$$P(t) = P_0 e^{-\beta t} + \frac{Z(K)}{\beta} \tag{3.97}$$

Thus $P(t) \to P^K$, and $P^K < P(t) < P_0$ for $P_0 > P^K$ and $P_0 < P(t) < P^K$ for $P_0 < P^K$. The equation for λ is

$$\dot{\lambda} = (\delta + \beta)\lambda - U_P(K, P(t)) > 0 \tag{3.98}$$

which can be solved as

$$\lambda(t) = \lambda(0)e^{(\delta+\beta)t} - \int_0^t e^{(\delta+\beta)(t-\tau)} U_P(K, P(\tau))d\tau =$$

$$= e^{(\delta+\beta)t}[\lambda(0) - \int_0^t e^{-(\delta+\beta)\tau} U_P(K, P(\tau))d\tau] \tag{3.99}$$

Let us denote the last integral above by $\zeta(t)$. It can be bounded as follows

$$\frac{1 - e^{-(\delta+\beta)t}}{\delta + \beta} \min\{U_P(K, P^K), U_P(K, P_0)\} \leq \zeta(t) \leq \frac{1 - e^{-(\delta+\beta)t}}{\delta + \beta} \max\{U_P(K, P^0), U_P(K, P^K)\} \tag{3.100}$$

Notice that on the left side of P^K (for $P_0 < P^K$) the value $\lambda(0)$ is always above the upper bound on $\zeta(t)$. This is so because of the conditions $\dot{\lambda} > 0$ and $U_P(P_0, K) > U_P(P^K, K)$ for $P_0 < P^K$.

Because the point (K, P^K) is not a stationary one, then from (3.84) we have

$$\frac{U_P(K, P^K)}{\delta + \beta} < -\frac{U_C(K, P^K)}{Z'(K)} = up(P^K) \tag{3.101}$$

that is λ^S is in the "feasible" region on the state-costate phase plane[4]. But $up(P^K)$ is nondecreasing. As to the right of P^K we have $U_P(P_0, K) < U_P(P^K, K)$, then all $\lambda(0)$ in the "boundary" region must also be above the upper boundary for $\zeta(t)$. Thus we see that all paths starting from the "boundary" $(C = K)$ region satisfy $\lambda(0) > \frac{\max\{U_P(K,P^0),U_P(K,P^K)\}}{\delta+\beta}$. By simple change of the time origin we can have the same condition for all paths which have entered the "boundary" region.

But for $\lambda(0) > \frac{\max\{U_P(K,P^0),U_P(K,P^K)\}}{\delta+\beta}$ we have $\lambda(t) > ae^{(\delta+\beta)t}$, where a is a constant, and therefore

$$\lim_{t\to\infty} e^{-\delta t} a e^{(\delta+\beta)t} = \infty \tag{3.102}$$

The problem is autonomous, the set of feasible consumptions $[0, K]$ is finite. The convex hull of the set of possible derivatives is $H = \{Z(C) - \beta P | C \in [0, K]\} = [Z(0) - \beta P, Z(K) - \beta P]$. For $P_0 < P^K$ we have $0 \in H$ and therefore all paths approaching the line $P = P^K$ from the left side are not optimal because they do not satisfy the condition (A.17).

As all paths considered are in the region where $C = K$, then the corresponding utility function is $U(K, P)$. However, $U_P < 0$, then all path to the right of the line $P = P^K$ give smaller value of the objective function than those to the left and therefore are also not optimal. This terminates the argument of nonoptimality of diverging paths.

Let us note that the path starting from the point $\lambda(0) = \frac{U_P(K,P^K)}{\delta+\beta}, P_0 = P^K$ stays there because for them $\dot{P} = \dot{\lambda} = 0$. However, for this path the necessary optimality conditions are not satisfied, and namely the condition (3.73), and the path staying there cannot be optimal.

The paths converging to the stationary solution may be only found in the isosectors II and III. They all approach the axis λ and when touching it "jump" to the point λ^P. Among them one comes directly to the point λ^P (denoted o on the Fig. 3.10). We show that it is optimal. Indeed, all paths above it finally diverge and were shown above to be nonoptimal. At the same time calculating the partial derivative of $U(C(\lambda, P), P)$ with respect to λ we get

$$U_\lambda = U_C C_\lambda > 0 \tag{3.103}$$

Thus, for all paths below o it is possible to choose a bigger value of λ to get a bigger value of the utility function. Then o must be optimal.

The Stationary Solution at $C = K$

The situation on the state-costate phase plane is presented on the Fig. 3.11. As the analysis of the changes of the signs of $\dot{P}$ and $\dot{\lambda}$ done for the equilibrium is still valid here, under discussion of differentiability in the previous section, we have $\dot{P} < 0$ to the right of the curve $\dot{P} = 0$ (isosectors II and III) and $\dot{P} > 0$ to the left of it (isosectors IV and I). We also have $\dot{\lambda} > 0$ above the curve $\dot{\lambda} = 0$ (isosectors IV and III) and $\dot{\lambda} < 0$ below it (isosectors I and II).

Now (P^K, λ^K) is above $up(P)$ (in the region of boundary points) and we see that is a saddle point. There exist only two paths converging to it, which correspond to the convergent solutions of $\lambda(t)$, shown to be impossible in the previous section. All other path diverge away. Those above diverge to the infinite point (P^K, ∞). As shown above, they are not optimal. Those below hit the line $\lambda = 0$ below λ^P. However, these points do not satisfy the necessary optimality conditions and therefore these paths cannot be optimal. Thus the only optimal point is (P^K, λ^S).

[4] Contrary to the assumption on the "boundary" solution which makes this point not to satisfy the necessary optimality conditions.

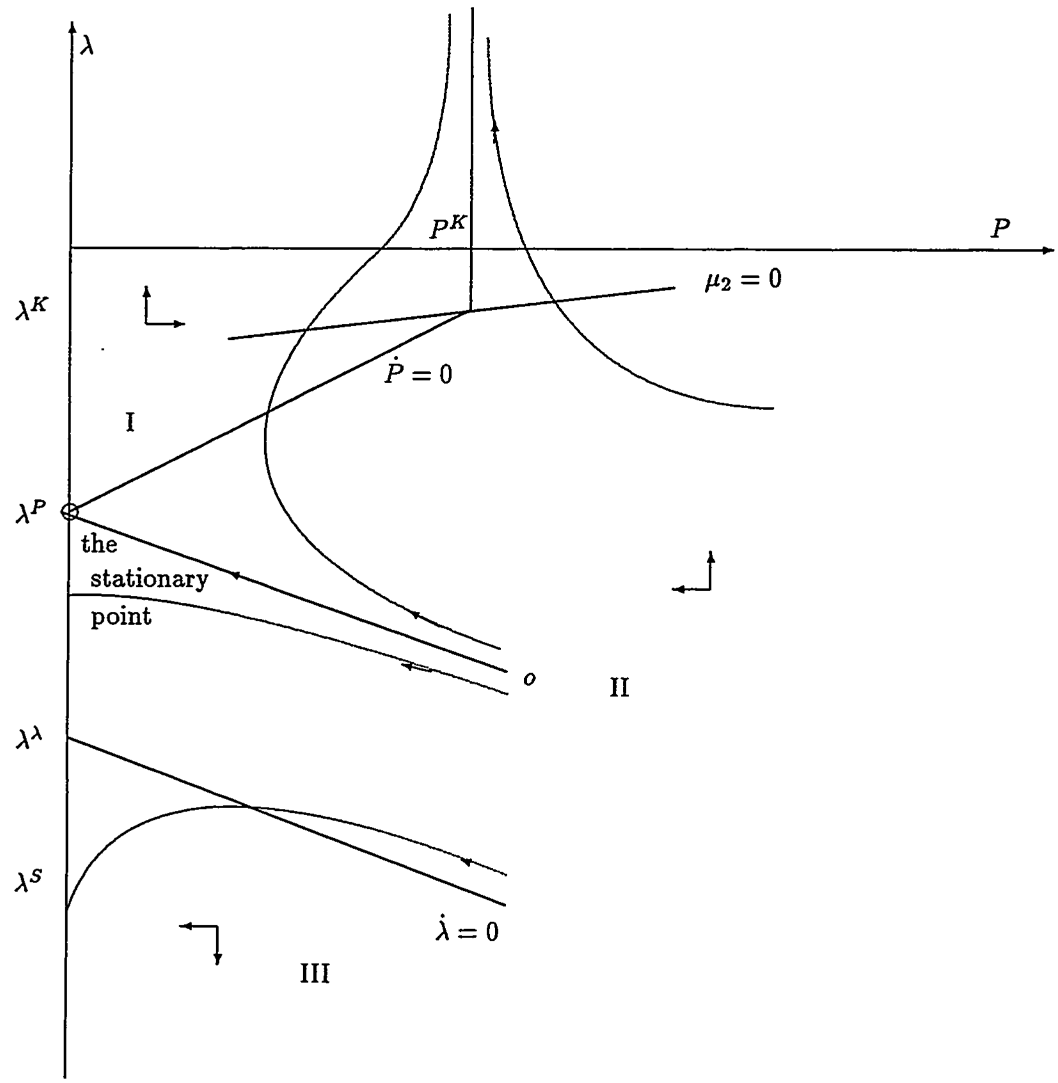

Figure 3.10: Analysis of the equations on the state-costate phase plane for the case when the stationary point is on the boundary $P = 0$.

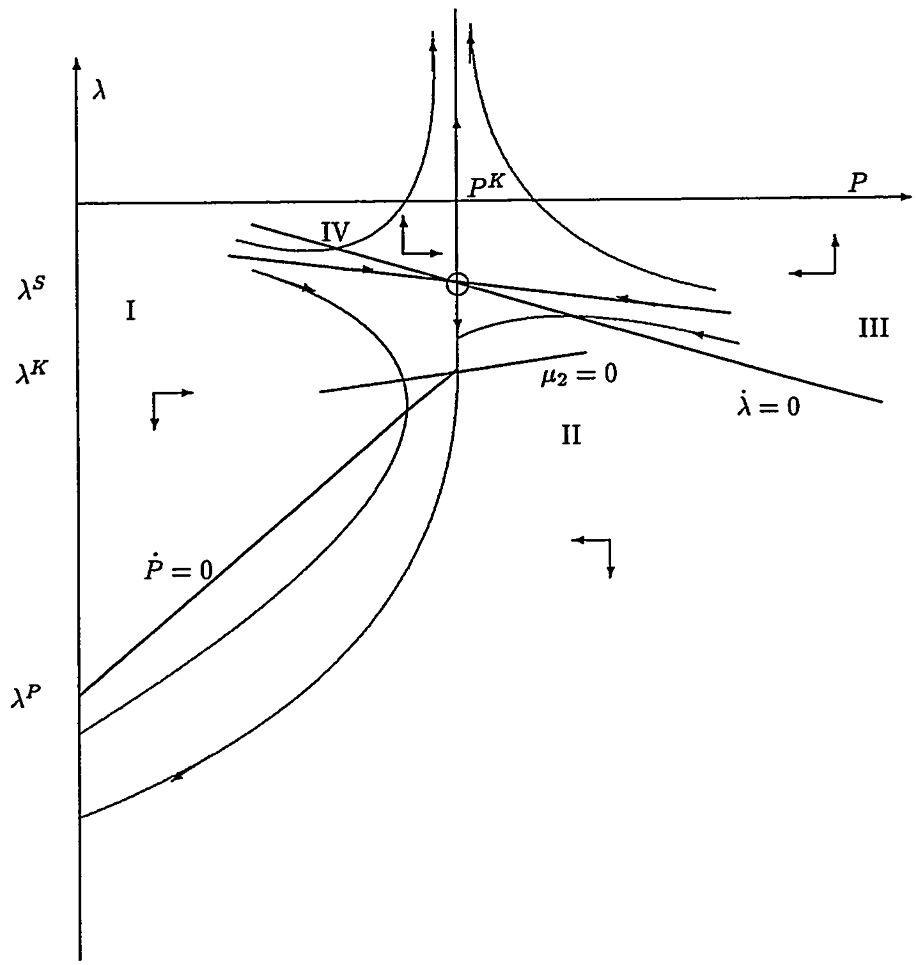

Figure 3.11: Analysis of the equations on the state-costate phase plane for the case when the stationary point is on the boundary $C = K$.

3.5 Summary of the One State Variable Problem

The general one state variable pollution problem (3.8) - (3.10) has always the unique optimal solution under assumptions taken. Depending on the problem function characteristic the optimal solution converges in time to either an equilibrium point in the interior of the set of feasible solutions, or to an equilibrium point on one of the boundaries $P = 0$ or $C = K$, or to an asymptotic stationary point on one of these boundaries. Besides the optimal paths converging to the optimal asymptotic solution (for the stationary point on a boundary $P = 0$ also some nonoptimal paths converge to the asymptotic optimal solution as well) there are nonoptimal paths which diverge to the nonoptimal asymptotic solution with the infinite costate variable or fall in the set of points unable to satisfy the necessary optimality conditions.

Chapter 4

Conclusions and Further Research

This report is closed in Chapter 3 with the analysis of the one state variable model. However, some two state variable models have also been analyzed in the literature. Section B.2 in Appendix B gives a short introduction to the basic theoretical tool which can be used there. Leaving apart the labour model there are three possible combinations of the pairs of different one state variable models among those listed in Section 2.1. Two of them include the pollution model. These pairs are: (1) pollution accumulation - capital accumulation models, and (2) pollution accumulation - resource extraction models. Pair (1) is a direct extension of the Forster one dimensional model discussed in the previous chapter, with the dynamic description of the capital accumulation (growth) replacing the static description of the Forster model.

Both pairs were discussed in the literature. Examples for the pair (1) may include [63], ch.5, sec.2, ex.3 (growth that pollutes), [42], [43] (which is also presented in [21], sce.15.2) or [77] and examples for the pair (1) [12], sec.4.33 (resource depletion and residual accumulation), [63], ch.3, sec.8, ex.12 (resource extraction with waste) or [72].

The models used there are usually of a simpler form than the one discussed in the previous chapter. Very often the utility function is separable in its arguments and the equations of motion are linear or at least separable and partly linear. Cases of more complicated formulation are either treated for the finite horizon ([63] where the products of arguments is present in the functions) or numerically (like in the Luptacik and Schubert model, see [21]). More general model problem was considered in [72] where Jacobian matrix analysis was used, as presented in Section B.2. These examples show increasing difficulty when a general theory and methods of analytical analysis are being developed for the multistate variable models.

Yet many real cases do involve much more than one or two state variables. Their analysis can now only be provided by numerical computations. With this respect the discrete time models are quite often considered, as more suitable for computer applications. There are, however, differences in application of continuous and discrete time models. For instance, the maximum principle does not always hold true for the discrete time problems, even when similar assumptions on the problem functions are taken, see Section A.2 in Appendix A. The solution of the nonlinear difference equations may be chaotic, even for quite simple functions, see e.g. [47] or [19], which is not a case for the continuous time analogues. Thus simplification in computations are to some extent counterweighted by more difficult analysis and uncertainty in the form of produced results.

There are more issues which were considered in the pollution control modelling literature. Examples include knowledge accumulation (or technology development) questions (e.g. [8]), transboundary pollution problems involving bargaining among many parties ([23]) or imperfect information and its influence on solutions obtained ([16], [69]). Many interesting issues can be also found in real system analysis, like in the case of abatement of CO_2 emission ([46], [54]).

Appendix A

The Maximum Principle

This Appendix aims at summarizing some basic results necessary for understanding the text of the report. No stress is put on proving the results or treating them with a particular mathematical rigour. For proofs, more results and economic interpretation see e.g. [21], [63], [35].

We assume throughout that the functions below have all continuous partial derivatives whenever necessary.

A.1 Continuous Time Problems

A.1.1 No State Constraints

Let us consider the following standard control problem

$$\text{maximize } \{\int_0^T U(x(t), u(t), t)dt + F(x(T), T)\} \tag{A.1}$$

subject to

$$\dot{x}(t) = f(x(t), u(t), t) \tag{A.2}$$

$$u(t) \in V, \qquad 0 \leq t \leq T$$

$$x(0) = x_0 \quad \text{(given)} \tag{A.3}$$

where $x(t)$ and $u(t)$ may be vector variables, and introduce *the Hamiltonian*

$$H(x(t), u(t), \eta(t), t) = U(x(t), u(t), t) + \eta(t) f(x(t), u(t), t) \tag{A.4}$$

where $\eta(t)$ is called the *adjoint* or *costate variable.* In the vector case the product of two vectors above and in the sequel is the scalar product. Then necessary conditions for a maximum (*the maximum principle*) are that optimal values $x^*(t), u^*(t), \eta^*(t)$ necessarily satisfy

$$H(x^*(t), u^*(t), \eta^*(t), t) = \max_{u(t) \in V} H(x^*(t), u(t), \eta^*(t), t)$$

$$\dot{\eta}^*(t) = -\frac{\partial H(x^*(t), u^*(t), \eta^*(t), t)}{\partial x(t)} \tag{A.5}$$

$$\eta^*(T) = \frac{dF(x^*(T), T)}{dx(T)}$$

If $F(x(T),T) \equiv 0$, then the last condition, called *the transversality condition* reduces to

$$\eta^*(T) = 0 \tag{A.6}$$

Usually the conditions above are given together with the initial constraints to form the closed set of equations for necessary solutions to the problem (A.1)

$$\dot{x}^*(t) = \frac{\partial H(x^*(t), u^*(t), \eta^*(t), t)}{\partial \eta(t)}$$

$$u^*(t) \in V, \quad 0 \leq t \leq T$$

$$x^*(0) = x_0$$

When an optimal solution is inside the constraint set V (this may be the case when there is no constraint on $u(t)$, i.e. $u(t)$ can take any value), then the maximum condition on Hamiltonian can be changed to the stationary condition

$$\frac{\partial H(x^*(t), u^*(t), \eta^*(t), t)}{\partial u(t)} = 0$$

In this case it is equivalent to the first order necessary conditions derived using the (linear) Lagrangian technique. Sometimes the terminal condition in (A.1) may be specified

$$x(T) \in X_T \tag{A.7}$$

Then the transversality condition (last of equations (A.5) or equation (A.6)) has to be substituted with more complicated conditions. Specifically, let:

1. $x(T) = x_T$ (fixed)
2. $x(T) \geq x_T$ (fixed)
3. $x(T)$ free

Then after redefining the Hamiltonian (which is at least necessary for cases 1 and 2 above) to

$$H(x^*(T), u^*(T), \eta^*(T), T) = \eta_0 U(x(t), u(t), t) + \eta(t) f(x(t), u(t), t) \tag{A.8}$$

we have, in addition to two first conditions in (A.5), the following transversality conditions:

1. $\eta^*(T)$ no condition
2. $\eta^*(T) \geq \eta_0^* \frac{dF(x^*(T),T)}{dx(T)}$ with $[\eta^*(T) - \eta_0^* \frac{dF(x^*(T),T)}{dx(T)}][x^*(T) - x_T] = 0$
3. $\eta^*(T) = \eta_0^* \frac{dF(x^*(T),T)}{dx(T)}$

and moreover

$$(\eta_0^*, \eta^*(t)) \neq (0,0), \quad \eta_0^* = 0 \text{ or } 1$$

In the vector case the inequalities above are component-wise (in fact also all three conditions above may be considered to be component-wise). Notice that condition 3 is actually the same as the transversality condition in (A.5), only adapted to the new definition (A.8) of the Hamiltonian.

One more class of problems of interest is connected with the free end time, when T is subject to optimization. This class of problems will not be considered here.

If we define *the value function*

$$J(x,t) = \max_{u(t)} \int_t^T U(x(t), u(t), t)dt$$

then, assuming J to be continuously differentiable, for the optimal solution there holds

$$\eta^*(t) = \frac{\partial J(x^*, t)}{\partial x}$$

This gives the marginal value for the state at time t. With this in mind the variable $\eta^*(t)$ is often called *the shadow price.*

In the economic literature *discounted* objection functions are often considered. Then the problem is

$$\text{maximize } \{\int_0^T U(x(t), u(t), t)e^{-\delta t}dt + F(x(T), T)e^{-\delta T}\} \tag{A.9}$$

subject to

$$\begin{aligned} &\dot{x}(t) = f(x(t), u(t), t) \\ &u(t) \in V, \quad 0 \leq t \leq T \\ &x(0) = x_0 \qquad \text{(given)} \end{aligned} \tag{A.10}$$

The Hamiltonian for this problem is

$$H(x(t), u(t), \eta(t), t) = U(x(t), u(t), t)e^{-\delta t} + \eta(t)f(x(t), u(t), t) \tag{A.11}$$

but it is common to define in this case a so called *current value Hamiltonian*

$$\tilde{H}(x(t), u(t), \lambda(t), t) = H(x(t), u(t), \eta(t), t)e^{\delta t} = U(x(t), u(t), t) + \lambda(t)f(x(t), u(t), t) \tag{A.12}$$

where $\lambda(t) = e^{\delta t}\eta(t)$. Now we have

$$\begin{aligned} \dot{\eta}^*(t) &= -\frac{\partial H(x^*(t), u^*(t), \eta^*(t), t)}{\partial x(t)} \\ &= -\frac{\partial U(x^*(t), u^*(t), t)}{\partial x(t)}e^{-\delta t} - \eta^*(t)\frac{\partial f(x^*(t), u^*(t), t)}{\partial x(t)} \end{aligned}$$

and

$$\dot{\eta}^*(t) = -\delta\lambda^*(t)e^{-\delta t} + \dot{\lambda}^* e^{-\delta t}$$

Inserting the latter to the former we get

$$\dot{\lambda}^*(t) - \delta\lambda^*(t) = -\frac{\partial \tilde{H}(x^*(t), u^*(t), \lambda^*(t), t)}{\partial x(t)}$$

It is easy to see that the other conditions are not affected by the change of the Hamiltonian definition and finally we get the set of necessary conditions for the problem (A.9)

$$\tilde{H}(x^*(t), u^*(t), \lambda^*(t), t) = \max_{u(t)\in V} \tilde{H}(x^*(t), u(t), \lambda^*(t), t)$$

$$\dot{\lambda}^*(t) - \delta\lambda^*(t) = -\frac{\partial \tilde{H}(x^*(t), u^*(t), \lambda^*(t), t)}{\partial x(t)}$$

$$\lambda^*(T) = \frac{dF(x^*(T),T)}{dx(T)} \tag{A.13}$$

$$\dot{x}^*(t) = \frac{\partial \tilde{H}(x^*(t),u^*(t),\lambda^*(t),t)}{\partial \lambda(t)}$$

$$u(t) \in V, \qquad 0 \le t \le T$$

$$x^*(0) = x_0$$

Simple calculations show that when an optimal solution is inside the constraint set V the maximum condition can also in this case be replaced by the stationary condition

$$\frac{\partial \tilde{H}(x^*(t),u^*(t),\lambda^*(t),t)}{\partial u(t)} = 0$$

Earlier conditions (A.7) and specification (A.8) for the problem also apply.

A.1.2 Infinite Horizon

Another interesting problem may be connected with the infinite horizon. Then the problem with the discounted objective function (A.9) changes to

$$\text{maximize} \{\int_0^\infty U(x(t),u(t),t)e^{-\delta t}dt\} \tag{A.14}$$

subject to

$$\dot{x}(t) = f(x(t),u(t),t) \tag{A.15}$$

$$u(t) \in V, \qquad t \ge 0$$

$$x(0) = x_0 \qquad \text{(given)}$$

This, of course, is connected with the requirement that the integral (A.14) is finite for all feasible control functions. (The case when this requirement is not fulfilled is considered in [63].) For the problem (A.9) only two first conditions of (A.5) are valid (for the Hamiltonian with η_0). Any asymptotic conditions of the type (A.7) require in this case more assumptions, see [21] or [63] for details.

However, if the problem is autonomous, i.e. U and f do not depend explicitly on t ($U_t = f_t = 0$) then the following condition applies

$$\lim_{t\to\infty} H(x^*(t),u^*(t),\eta(t)) = \lim_{t\to\infty} e^{-\delta t}\tilde{H}(x^*(t),u^*(t),\lambda(t)) = 0 \tag{A.16}$$

Moreover, if $U \ge 0$ for all admissible $x(t)$ and $u(t)$ (or V is a finite set), and zero is in the interior of the convex hull of the set of possible speeds for the admissible controls, i.e. $0 \in \text{int co}\{f(x^*,u)|u \in V\}$ for sufficiently big values of t, then

$$\lim_{t\to\infty} e^{-\delta t}\lambda(t) = 0 \tag{A.17}$$

An additional condition can be formulated if the function $H^0(x,\eta,t) = \max_{u\in V} H(x,u,\eta,t)$ is concave in x, for any η and t. Then for any admissible $x(t)$

$$\lim_{t\to\infty} e^{-\delta t}\lambda(t)[x(t) - x^*(t)] \ge 0 \tag{A.18}$$

provided $\eta(t)$ exists.

A.1.3 State Constraints

In some problems *state constraints* may be necessary. Then general form of this kind of problem might take the form

$$\text{maximize}\ \{\int_0^T U(x(t),u(t),t)dt + F(x(T),T)\} \tag{A.19}$$

subject to

$$\begin{gathered}\dot{x}(t) = f(x(t),u(t),t) \\ g(x(t),u(t),t) \geq 0, \qquad 0 \leq t \leq T \\ x(0) = x_0 \qquad \text{(given)}\end{gathered} \tag{A.20}$$

In this case, for the maximum principle to hold, *a constraint qualification* must be satisfied. It can take the form that the following compound matrix

$$[\frac{\partial g(x^*(t),u^*(t),t)}{\partial u(t)} \quad diag(g(x^*(t),u^*(t),t))] \tag{A.21}$$

has the full rank, where $diag(g(x^*(t),u^*(t),t))$ is a square matrix with zeros except the main diagonal, where elements of the vector $g(x^*(t),u^*(t),t)$ are placed. This condition can also equivalently be written that the matrix $\frac{\partial g_E(x^*(t),u^*(t),t)}{\partial u(t)}$ has full rank, where $g_E(x^*(t),u^*(t),t)$ is the vector of active constraints in an optimal solution, i.e. it contains these and only these constraints which satisfy $g_E(x^*(t),u^*(t),t)=0$. In other words, this can be formulated that the derivatives, with respect to $u(t)$, of active constraints at an optimal solution $x^*(t),u^*(t)$ be linearly independent. Let us additionally define *the Lagrange function* L, with the Hamiltonian definition (A.8), and *the set of feasible controls* Ω as

$$\begin{gathered}L(x(t),u(t),\eta(t),\mu(t),t) = H(x(t),u(t),\eta(t),t) + \mu(t)g(x(t),u(t),t) \\ \Omega(x(t),t) = \{u(t) \mid g(x(t),u(t),t) \geq 0\}\end{gathered} \tag{A.22}$$

then an optimal solution $x^*(t),u^*(t),\eta_0^*,\eta^*(t),\mu^*(t)$, satisfying a constraint qualification, necessarily satisfy

$$\begin{gathered}H(x^*(t),u^*(t),\eta^*(t),t) = \max_{u(t)\in\Omega(x^*(t),t)} H(x^*(t),u(t),\eta^*(t),t) \\ \dot{\eta}^*(t) = -\frac{\partial L(x^*(t),u^*(t),\eta^*(t),\mu^*(t),t)}{\partial x(t)} \\ \frac{\partial L(x^*(t),u^*(t),\eta^*(t),\mu^*(t),t)}{\partial u(t)} = 0 \\ (\eta_0^*,\eta^*(t)) \neq (0,0), \quad \eta_0^* = 0 \text{ or } 1 \\ \mu^*(t) \geq 0, \quad \mu^*(t)g(x^*(t),u^*(t),t) = 0 \quad \text{(complementary slackness)} \\ g(x(t),u(t),t) \geq 0, \qquad 0 \leq t \leq T \\ \eta^*(T) = \eta_0^*\frac{dF(x^*(T),T)}{dx(T)} \\ x^*(0) = x_0\end{gathered} \tag{A.23}$$

Similarly as before, if $F(x(T),T) \equiv 0$, then the last but one condition (the transversality condition) reduces to

$$\eta^*(T) = 0$$

and with the end conditions

$$a(x(T),T) \geq 0, \quad b(x(T),T) = 0$$

the transversality condition becomes

$$\eta^*(t) = \eta_0^* \frac{dF(x^*(T),T)}{dx(T)} + \alpha \frac{\partial a(x^*(T),T)}{\partial x(T)} + \beta \frac{\partial b(x^*(T),T)}{\partial x(T)}$$

with the complementary slackness condition

$$\alpha \geq 0, \quad \alpha a(x^*(T),T) = 0$$

Let us notice that this is actually a more general form of conditions given after (A.7).

The discounted case also can be treated similarly as before, resulting in a change of the condition for the adjoint variable

$$\dot{\lambda}^*(t) - \delta\lambda = -\frac{\partial L(x^*(t), u^*(t), \lambda^*(t), \mu^*(t), t)}{\partial x(t)} \tag{A.24}$$

other conditions being of the same form.

A.2 Discrete Time Problems

Some problems in pollution control can be more conveniently formulated as a discrete time problems. The discrete time analogue of the problem (A.1) - (A.3) will be

$$\text{maximize} \left\{\sum_{n=0}^{N-1} U_n(x_n, u_n) + F(x_N)\right\} \tag{A.25}$$

subject to

$$x_{n+1} = f_n(x_n, u_n), \qquad n = 0, 1, \ldots, N-1 \tag{A.26}$$

$$u_n \in V_n, \quad n = 0, 1, \ldots, N-1$$

$$x_0 \quad \text{(given)}$$

Provided u_n is not constrained or an optimal solution is inside the constraint set, by simple Lagrange multiplier reasoning a set of equations necessary for a solution to be optimal can be obtained. Defining *the Hamiltonian*

$$H_n(x_n, u_n) = U_n(x_n, u_n) + \eta_{n+1} f_n(x_n, u_n) \tag{A.27}$$

the stationary solution satisfies the following equations for $n = 0, 1, \ldots, N-1$

$$\begin{aligned} \frac{\partial H_n(x_n^*, u_n^*)}{\partial u_n} &= 0 \\ \eta_n^* &= \frac{\partial H_n(x_n^*, u_n^*)}{\partial x_n} \\ x_{n+1}^* &= f_n(x_n^*, u_n^*) \\ u_n^* &\in V_n \\ \eta_N^* &= \frac{dF(x_N^*)}{dx_N} \end{aligned} \tag{A.28}$$

$$x_0^* = x_0 \quad \text{(given)}$$

Similarly as in the continuous time for $F(x_N) \equiv 0$ the last but one (transversality) condition changes to

$$\eta_N^* = 0$$

and for a specified end value condition it is substituted with more complicated conditions.

However, unlike in the continuous time, now the first equation cannot be simply changed to the maximum of the Hamiltonian over V_n with respect to u_n. For this an additional assumption of the *directional convexity* must be satisfied [30], [31], [7]. Those problems which do not possess this property may be approached by using *the generalized maximum principle* [53]. In order not to complicate the presentation this material will not be given here.

For a *discounted* problem we have

$$\text{maximize } \{\sum_{n=0}^{N-1} \rho^n U_n(x_n, u_n) + \rho^N F(x_N)\} \tag{A.29}$$

subject to

$$\begin{gathered} x_{n+1} = f_n(x_n, u_n), \qquad n = 0, 1, \ldots, N-1 \\ u_n \in V_n, \qquad n = 0, 1, \ldots, N-1 \\ x_0 \quad \text{(given)} \end{gathered} \tag{A.30}$$

where ρ is *the discount factor*. Sometimes also *a discount rate* δ is used. They are connected by the equation $\rho = \frac{1}{1+\delta}$. Similarly as in the previous subsection we can define the new costate variable $\lambda_n = \rho^{-n}\eta_n$ and introduce *the current value Hamiltonian* as

$$\tilde{H}_n(x_n, u_n) = \rho^{-n} H_n(x_n, u_n) = U_n(x_n, u_n) + \rho\lambda_{n+1} f_n(x_n, u_n) \tag{A.31}$$

which then lead to the necessary conditions for $n = 0, 1, \ldots, N-1$

$$\begin{gathered} \frac{\partial \tilde{H}_n(x_n^*, u_n^*)}{\partial u_n} = 0 \\ \lambda_n^* = \frac{\partial \tilde{H}_n(x_n^*, u_n^*)}{\partial x_n} \\ x_{n+1}^* = f_n(x_n^*, u_n^*) \\ u_n^* \in V_n \\ \lambda_N^* = \frac{dF(x_N^*)}{dx_N} \\ x_0^* = x_0 \quad \text{(given)} \end{gathered} \tag{A.32}$$

Appendix B

Phase Plane Analysis

B.1 One-Dimensional State

In the economic literature the infinite horizon problem is often considered. The appropriate maximum principle equations for this case were presented shortly in Appendix A. We discuss now this case in more detail. To allow the graphical presentation and also to make the analysis simpler we assume now that $x(t)$ and $u(t)$ are *scalar functions*. Moreover, we assume that all functions are *twice continuously differentiable.*

B.1.1 Problem Formulation

Let us consider then the following control problem with discounting

$$\text{maximize } \{\int_0^\infty U(x(t), u(t))e^{-\delta t}dt\} \tag{B.1}$$

subject to

$$\dot{x}(t) = f(x(t), u(t))$$
$$x(0) = x_0 \ (given)$$

Note that the functions do not depend on t and there is no restriction neither on $x(t)$ nor $u(t)$. The maximum principle conditions are now as follows

$$\frac{\partial \tilde{H}(x^*(t), u^*(t), \lambda^*(t))}{\partial u(t)} = 0$$

$$\dot{\lambda}^*(t) - \delta\lambda^*(t) = -\frac{\partial \tilde{H}(x^*(t), u^*(t), \lambda^*(t), t)}{\partial x(t)}$$

$$\dot{x}^*(t) = \frac{\partial \tilde{H}(x^*(t), u^*(t), \lambda^*(t), t)}{\partial \lambda(t)}$$

$$x^*(0) = x_0$$

which, from the current value Hamiltonian definition, can be written also as

$$\frac{\partial U(x^*(t), u^*(t))}{\partial u(t)} + \lambda^*(t)\frac{\partial f(x^*(t), u^*(t))}{\partial u(t)} = 0$$

$$\dot{\lambda}^*(t) - \delta\lambda^*(t) = -\frac{\partial U(x^*(t), u^*(t))}{\partial x(t)} - \lambda^*(t)\frac{\partial f(x^*(t), u^*(t))}{\partial x(t)} \tag{B.2}$$
$$\dot{x}^*(t) = f(x^*(t), u^*(t))$$
$$x^*(0) = x_0$$

B.1.2 An Equilibrium

We would like now to examine how a solution is evolving in time and specifically for long times (asymptotically). It may be expected that in many real life problems the functions should stabilize in time (*a steady state solution*). In particular we see that any constant functions (x^*, u^*, λ^*) satisfying

$$\dot{x}^*(t) = \dot{u}^*(t) = \dot{\lambda}^*(t) = 0$$

may be solutions of the system of equations. Any point satisfying the above conditions is also called *a critical point* or *an equilibrium*. The maximum conditions at the equilibrium reduce now to

$$\frac{\partial U(x^*, u^*)}{\partial u} + \lambda^* \frac{\partial f(x^*, u^*)}{\partial u} = 0$$
$$(\delta - \frac{\partial f(x^*, u^*)}{\partial x})\lambda^* - \frac{\partial U(x^*, u^*)}{\partial x} = 0 \tag{B.3}$$
$$f(x^*, u^*) = 0$$

These form three algebraic equations with three unknowns. Provided that $\frac{\partial f(x^*,u^*)}{\partial u} \neq 0$ we get from the first one

$$\lambda^* = -\frac{\frac{\partial U(x^*,u^*)}{\partial u}}{\frac{\partial f(x^*,u^*)}{\partial u}} \tag{B.4}$$

Substituting the above into the second equation in (B.3) and assuming additionally that $\frac{\partial U(x^*,u^*)}{\partial u} \neq 0$ gives

$$\frac{\partial f(x^*, u^*)}{\partial x} - \frac{\frac{\partial U(x^*,u^*)}{\partial x}}{\frac{\partial U(x^*,u^*)}{\partial u}} \frac{\partial f(x^*, u^*)}{\partial u} = \delta \tag{B.5}$$

Together with the third equation in (B.3)

$$f(x^*, u^*) = 0$$

they may (possibly) give solutions for the steady state values (x^*, u^*) which can then be used to find the solution for λ^* from (B.4). Let us summarize also the assumptions taken

$$\frac{\partial f(x^*, u^*)}{\partial u} \neq 0 \quad \text{and} \quad \frac{\partial U(x^*, u^*)}{\partial u} \neq 0 \tag{B.6}$$

B.1.3 State-Control and State-Costate Differential Equations

To find the asymptotic behaviour of a solution we have to analyze again the system of equations (B.2). However, to present the solution on a plane we have to deal only with two differential equations. Thus one of three unknown functions must be eliminated. Out of three possibilities usually two are considered: (1) eliminate $\lambda^*(t)$, then deal with the state–control case $(x^*(t), u^*(t))$, (2) eliminate $u^*(t)$, then deal with the state–costate case $(x^*(t), \lambda^*(t))$. The trajectories of these solutions can be drawn on planes (x^*, u^*) or (x^*, λ^*), respectively. They are called *the phase planes.* We start analysis with the former.

To simplify notation we drop now all the arguments except time and denote the derivatives with respect to non–time arguments by a subscript. This allows us to write the system of equations (B.2), after dropping the last one, which is not relevant for the asymptotic analysis, as

$$\begin{gathered} U_u + \lambda^*(t) f_u = 0 \\ \dot{\lambda}^*(t) - \delta\lambda^*(t) = -U_x - \lambda^*(t) f_x \\ \dot{x}^*(t) = f \end{gathered} \tag{B.7}$$

Now, under assumption that $f_u \neq 0$ at $(x^*(t), u^*(t))$, from the first equation we get

$$\lambda^*(t) = -\frac{U_u}{f_u} \tag{B.8}$$

and differentiating the same equation with respect to time

$$U_{ux}\dot{x}^*(t) + U_{uu}\dot{u}^*(t) + \dot{\lambda}^*(t) f_u + \lambda^*(t)(f_{ux}\dot{x}^*(t) + f_{uu}\dot{u}^*(t)) = 0$$

Substituting now for $\lambda^*(t)$ from the former and for $\dot{x}^*(t)$ from the third equation in (B.7) yields

$$(U_{uu} - U_u\frac{f_{uu}}{f_u})\dot{u}^*(t) + (U_{ux} - U_u\frac{f_{ux}}{f_u})f + \dot{\lambda}^*(t) f_u = 0$$

Substituting now for $\lambda^*(t)$ in the second equation in (B.7) we have

$$\dot{\lambda}^*(t) = -U_x - \frac{U_u}{f_u}(\delta - f_x) \tag{B.9}$$

Now, from both the above we get

$$(U_{uu} - U_u\frac{f_{uu}}{f_u})\dot{u}^*(t) + (U_{ux} - U_u\frac{f_{ux}}{f_u})f - U_x f_u - U_u(\delta - f_x) = 0 \tag{B.10}$$

If the expression before $\dot{u}^*(t)$ is nonzero, then by introduction of a suitable function h this can be written as the first equation below. Then also the second equation is rewritten from (B.2) to complete the system of equations.

$$\dot{u}^*(t) = h(x^*(t), u^*(t)) \tag{B.11}$$

$$\dot{x}^*(t) = f(x^*(t), u^*(t))$$

These are the equations for the state–control phase plane analysis. Note that during the derivation we assumed that at $(x^*(t), u^*(t))$ the following holds

$$f_u \neq 0 \quad \text{and} \quad U_{uu} - U_u\frac{f_{uu}}{f_u} \neq 0 \tag{B.12}$$

Note also that with the assumptions taken both functions on the right side in (B.11) are twice continuously differentiable.

For the second case, i.e. state–costate equations, we assume that the first equation in (B.7) can be solved for $u^*(t)$, at least locally. For this we have to have at $(x^*(t), u^*(t))$

$$U_{uu} + \lambda^*(t) f_{uu} \neq 0$$

Note that because of (B.8) the above is equivalent to the second condition of (B.12). Then from the implicit function theorem we know that there exists a function $u^*(t) = \hat{u}(x^*(t), \lambda^*(t))$ that satisfies the first equation of (B.7) identically. Moreover, this function is twice continuously differentiable. Inserting $\hat{u}$ in the second and third equation of (B.7) we get a system of two differential equations, which by introduction of suitable functions q and g may be written as

$$\begin{aligned} \dot{\lambda}^*(t) &= q(x^*(t), \lambda^*(t)) \\ \dot{x}^*(t) &= g(x^*(t), \lambda^*(t)) \end{aligned} \tag{B.13}$$

The functions q and g are twice continuously differentiable, as well.

B.1.4 Classification of Steady-State Solutions

To proceed further we need to recall some facts from the theory of ordinary differential equations, see e.g. [11] or [5]. To focus attention we concentrate on equations (B.11). The right hand sides can be linearized around a critical point giving

$$\begin{aligned} \dot{x}^*(t) &= f_x(x^*(t), u^*(t))(x^*(t) - x^*) + f_u(x^*(t), u^*(t))(u^*(t) - u^*) + O(r^2) \\ \dot{u}^*(t) &= h_x(x^*(t), u^*(t))(x^*(t) - x^*) + h_u(x^*(t), u^*(t))(u^*(t) - u^*) + O(r^2) \end{aligned} \tag{B.14}$$

where $r = \sqrt{(x^*(t) - x^*)^2 + (u^*(t) - u^*)^2}$. We assume that $f_x h_u - f_u h_x \neq 0$, as otherwise these linear parts would be linearly dependent. Then (x^*, u^*) is the only critical point of the linear system of equations.

Consider now the asymptotic behaviour of the linear part of the system (B.14). The solution of this system depends on the eigenvalues of the following Jacobian matrix

$$J = \begin{bmatrix} f_x & f_u \\ h_x & h_u \end{bmatrix}$$

where again the arguments $(x^*(t), u^*(t))$ have been dropped to simplify notation. Notice that with the assumptions taken above the Jacobian matrix J is nonsingular. The eigenvalues can be found by solving the characteristic equation

$$\det[J - sI] = \begin{vmatrix} f_x - s & f_u \\ h_x & h_u - s \end{vmatrix} = s^2 - (f_x + h_u)s + (f_x h_u - f_u h_x) = 0$$

The eigenvalues depend on the determinant

$$\Delta = (f_x + h_u)^2 - 4(f_x h_u - f_u h_x) = (f_x - h_u)^2 + 4 f_u h_x$$

and are given by

$$s_{1/2} = \frac{1}{2}(f_x + h_u \pm \sqrt{\Delta})$$

Let us also notice that from the Vieta formulae we have

$$s_1 \cdot s_2 = f_x h_u - f_u h_x = \det J \tag{B.15}$$

Thus if $\det J < 0$, then $\Delta > 0$ and the solutions are real and of the opposite signs.

The eigenvalues may be classified as follows ([11], Chapter 15):

(1) real and distinct when $\Delta > 0$,

(2) complex conjugate when $\Delta < 0$, and

(3) double real when $\Delta = 0$.

Now, the equilibrium of the nonlinear system of equations can be classified as

(i) an unstable node if both eigenvalues are real and $s_1, s_2 > 0$,

(ii) a stable node if both eigenvalues are real and $s_1, s_2 < 0$,

(iii) a saddle point if both eigenvalues are real and $s_1 \cdot s_2 < 0$,

(iv) a stable spiral if both eigenvalues are complex and their real part is negative,

(v) an unstable spiral if both eigenvalues are complex and their real part is positive,

(vi) a center (with solutions on circles) if both eigenvalues are complex and their real part is zero; then the asymptotic solutions are periodic, i.e. stable but not strictly stable.

In the cases (ii) and (iv) any solution starting in the neighbourhood of the equilibrium will converge to it. In the cases (i) and (v) any solution starting in the neighbourhood of the equilibrium will diverge out of it. In the case (vi) any solution will tend to a periodic steady state function, which is represented as a circle on the phase plane. In the case (iii) the solution may be convergent or divergent.

The solution of the linearized equation is of the form

$$\begin{bmatrix} x^*(t) - x^* \\ u^*(t) - u^* \end{bmatrix} = aW_1 e^{s_1 t} + bW_2 e^{s_2 t} \tag{B.16}$$

where W_1 and W_2 are the eigenvectors of the linearized system, given by the equation

$$(J - s_i I)W_i = 0, \qquad i = 1, 2 \tag{B.17}$$

Notice that because s_i is an eigenvector of the matrix J, then the matrix $J - s_i I$ is a singular matrix and therefore the equation (B.17) has nonzero solutions for W_i (they actually all lie on a halfline for each s_i). Specifically, the steady state solutions for the case (1) above (viz., $\Delta > 0$) are of the form

$$v_i(t) = c_{i1} e^{s_1 t} + c_{i2} e^{s_2 t} \qquad i = 1, 2$$

where $v_i(t)$ is either $x^*(t) - x^*$ for $i = 1$ or $u^*(t) - u^*$ for $i = 2$. For the case (2) it is of the form

$$v_i(t) = [c_{i1} \cos(\beta t) + c_{i2} \sin(\beta t)] e^{\alpha t} \qquad i = 1, 2$$

where α is a real part and β is an imaginary part of the complex eigenvalues. Finally, for the case (3) the steady state solution is of the form

$$v_i(t) = [c_{i1} + c_{i2} t] e^{st} \qquad i = 1, 2$$

as in this case $s_1 = s_2 = s$.

We now return to the nonlinear system of differential equations. Under the assumptions on twice continuous differentiability and nonsingularity of the Jacobian matrix J, the point (x^*, u^*) is an isolated critical point, i.e. there exist a circle around it in which there is no other critical point. Moreover, the following results are true ([11], Chapter 15):

1. stable nodes and spirals of the linear system correspond to stable nodes and spirals of the nonlinear system, respectively,
2. a center of the linear system corresponds to a center or a spiral (stable or unstable) of the nonlinear system,
3. a saddle point of the linear system corresponds to a saddle point of the nonlinear system; moreover in this case there exist only two orbits converging to the equilibrium, any other orbits tend away from them as $t \to \infty$; there exist also two and only two orbits diverging from the equilibrium, any other orbit tends to them as $t \to \infty$.

Intuitively the last case is obvious when we take into consideration (iv) and the solution for (1). If $s_1 < 0 < s_2$, then the converging solutions are those, for which $c_{i2} = 0$ and the diverging those, for which $c_{i1} = 0$.

B.1.5 Phase Planes

Given a problem at hand the asymptotic solution may then be found as described above. Moreover, the analysis of the problem can be conveniently done on a phase plane. Before presenting this possibility we need some assumptions. In the economic models some "standard" set of assumptions on functions U and f is usually taken. We give such two sets of assumptions below, but other variants are also possible. Basically, (1) below pertains to the resource exploitation models and (2) to pollution control models.

Assumptions connected with *the utility (or cost) function* U:

(U1) (1) $U_x > 0$ (U increasing in x), or (2) $U_x < 0$ (U decreasing in x),

(U2) $U_{xx} \leq 0$ (U concave in x),

(U3) (1) $U_u < 0$ (U decreasing in u), or (2) $U_u > 0$ (U increasing in u),

(U4) $U_{uu} \leq 0$ (U concave in u),

(U5) $U_{ux} \leq 0$.

In resource exploitation control models x may be the resource stock and u the effort of its exploitation. This gives intuitive meaning to assumptions (1) in (U1) and (U3). In pollution control models x may be pollution and u consumption which gives intuitive meaning to assumption (2) in (U1) and (U3). Assumptions on concavity (U2) and (U4) are often regarded as *risk aversion* of the planner.

Assumptions connected with *the equation of motion function* f:

(M1) (1) $f_x < 0$ (f decreasing in x), or (2) $f_x > 0$ (f increasing in x),

(M2) $f_{xx} \leq 0$ (f concave in x),

(M3) (1) $f_u > 0$ (f increasing in u), or (2) $f_u < 0$ (f decreasing in u),

(M4) $f_{uu} \leq 0$ (f concave in u),

(M5) $f_{ux} \leq 0$.

With the interpretation as above (M3) (1) says that the rate of the resource stock exploitation increases with bigger effort and (M3) (2) says that rate of pollution stock accumulation increases with bigger consumption. Other are just reasonable convenient assumptions not contradicting the intuition.

Moreover, the following additional assumption is usually made:

(C) $U_{uu} + f_{uu} < 0$ (at least one of the functions U or f is strictly concave in u).

Let us check that with these new set of assumptions our old assumptions are satisfied. The first condition in (B.12) follows directly from (M3). The second condition in (B.12) is implied by (U4), (M3), (M4) and (C). Also assumptions (B.6) are obviously satisfied. Moreover

$$H_{uu} = U_{uu} + \lambda^*(t) f_{uu}$$

As from (B.4) $\lambda^*(t) > 0$ then the Hamiltonian is concave in $u(t)$. This means that its stationary point, if exists, is unique.

The State-Costate Phase Plane

We analyze now the state-costate phase plane. We start with the partial derivatives of the implicit function $\hat{u}(x^*(t), \lambda^*(t))$ solving the first equation of (B.7). Calculating the partial derivatives with respect to $x^*(t)$ and $\lambda^*(t)$, respectively, we have

$$U_{uu}\hat{u}_x + U_{ux} + \lambda^*(t)(f_{uu}\hat{u}_x + f_{ux}) = 0$$

Then

$$\hat{u}_x = -\frac{U_{ux} + \lambda^*(t) f_{ux}}{U_{uu} + \lambda^*(t) f_{uu}} \leq 0 \tag{B.18}$$

as from (B.8), (U3) and (M3) we have $\lambda^*(t) > 0$. Similarly we get

$$\hat{u}_\lambda = -\frac{f_u}{U_{uu} + \lambda^*(t) f_{uu}} \begin{cases} > 0 & \text{for (1)} \\ < 0 & \text{for (2)} \end{cases} \tag{B.19}$$

On the phase plane the critical point can be found as an intersection of two *isoclines* $\dot{x}^*(t) = 0$ and $\dot{\lambda}^*(t) = 0$, see Fig. B.1. Let us show that the intersection point is unique. Consider first the case $\dot{x} = 0$. We look for the derivative, at the point x^*, of the implicit function $\lambda_1(x)$ satisfying the last equation of (B.7) for $\dot{x}^*(t) = 0$, i.e the equation

$$f(x, \hat{u}(x, \lambda_1(x))) = g(x, \lambda_1(x)) = 0$$

Differentiating with respect to x

$$f_x + f_u(\hat{u}_x + \hat{u}_\lambda \frac{d\lambda_1(x)}{dx}) = 0$$

and taking into account (B.18) and (B.19) we have

$$\frac{d\lambda_1(x)}{dx} = -\frac{1}{\hat{u}_\lambda}(\hat{u}_x + \frac{f_x}{f_u}) \begin{cases} > 0 & \text{for (1)} \\ < 0 & \text{for (2)} \end{cases} \tag{B.20}$$

Similarly for the function $\lambda_2(x)$ satisfying the equation for $\dot{\lambda}(t) = 0$, i.e. from the second of equations (B.7)

$$[\delta - f_x(x, \hat{u}(x, \lambda_2(x))]\lambda_2(x) = U_x(x, \hat{u}(x, \lambda_2(x)) \tag{B.21}$$

Differentiating with respect to x we get

$$-(f_{xx} + f_{xu}\hat{u}_\lambda \frac{d\lambda_2(x)}{dx})\lambda_2(x) + (\delta - f_x)\frac{d\lambda_2(x)}{dx} = U_{xx} + U_{xu}\hat{u}_\lambda \frac{d\lambda_2(x)}{dx}$$

or

$$\frac{d\lambda_2(x)}{dx} = \frac{U_{xx} + \lambda^* f_{xx}}{\delta - f_x - \hat{u}_\lambda(U_{xu} - \lambda^* f_{xu})} \begin{cases} < 0 & \text{for (1)} \\ > 0 & \text{for (2)} \end{cases} \tag{B.22}$$

where the following inequalities were used

$$\delta - f_x \begin{cases} > 0 & \text{for (1)} \\ < 0 & \text{for (2)} \end{cases} \tag{B.23}$$

(see (B.21)) and $\lambda^* > 0$ (from (B.8)). Then for the case (1) the curve $\lambda_1(x)$ is increasing and $\lambda_2(x)$ decreasing. So they cut in one point. For the case (2) the slopes change but are also of the opposite sign. So also in this case the curves cut in one point. Notice, that the formulae (B.20) and (B.22) allow us to calculate the directions of the solution trajectories (orbits) going to or from the equilibrium.

Now, the isoclines $\dot{x}(t) = 0$ and $\dot{\lambda}(t) = 0$ divide the positive orthant into four *isosectors* labeled on the figure I, II, III and IV. Let us consider the behaviour of the orbits in different isosectors. Let us begin with the isocline $\dot{x}(t) = 0$. For a constant λ we have (recall from (B.13) that $\dot{x} = g$)

$$g_x = f_x \begin{cases} < 0 & \text{for (1)} \\ > 0 & \text{for (2)} \end{cases}$$

As $g = 0$ for $x = x^*$, from the above we conclude that

$$\dot{x} \begin{cases} < 0 & \text{for (1)} \\ > 0 & \text{for (2)} \end{cases}$$

So with the growing time the horizontal direction of the move of the point on the orbits above the curve $\dot{x}(t) = 0$ (i.e. in the isosectors I and IV) is leftwards for the case (1) and rightwards for the case (2), and below the curve (in the isosectors II and III) in opposite directions. This is visualized by appropriate arrows in the isosectors on the Fig. B.1 (for the case (2), but similar drawing can be made for the case (1)).

Let us consider now the isocline $\dot{\lambda}(t) = 0$. For a constant x we have from (B.9) (recall also from (B.13) that $\dot{\lambda} = q$)

$$q_\lambda = -U_{xu}\hat{u}_\lambda + \delta - f_x - \lambda f_{xu}\hat{u}_\lambda \begin{cases} > 0 & \text{for (1)} \\ < 0 & \text{for (2)} \end{cases}$$

(Note that for (2) from (B.23) $\delta - f_x < 0$ for $\dot{\lambda}(t) = 0$, then because of continuity this must be true in some neigbourhood of this line.) On the same reason as above the vertical direction of the point above the curve $\dot{\lambda}(t) = 0$ (in the isosectors I and II) is upwards for the case (1) and downwards for the case (2), and below the curve (in the isosectors III and IV) in opposite directions.

Thus looking at the Fig. B.1 (for the case (2)) we see that the equilibrium is a saddle point. There are two orbits (o_1 and o_3) converging to the equilibrium, two (o_2 and o_4) diverging from it, and the rest tends away from o_1 or o_3 to o_2 or o_4.

This elementary analysis will be now followed by the analysis of the Jacobian matrix. We have

$$\det J = \begin{bmatrix} g_x & g_\lambda \\ q_x & q_\lambda \end{bmatrix} = g_x q_\lambda - g_\lambda q_x$$

The derivatives g_x and q_λ were already calculated. For the other we have

$$g_\lambda = \frac{\partial}{\partial \lambda} f(x, \hat{u}(x, \lambda)) = f_u \hat{u}_\lambda > 0$$

$$q_x = -U_{xx} - \lambda^*(t) f_{xx} \geq 0$$

then $\det J < 0$ which means that the eigenvalues are real and of opposite signs (recall (B.15)). Thus the equilibrium is a saddle point.

Let us notice that the directions of the orbits converging to and diverging from the equilibrium, besides of calculating the slopes of $\lambda_1(x)$ and $\lambda_2(x)$, can be also found by calculating eigenvectors of the Jacobian matrix J. If we have $s_1 < 0 < s_2$, then in (B.16) W_1 and $-W_1$ are the directions of the convergent orbits and W_2 and $-W_2$ are those of the divergent ones.

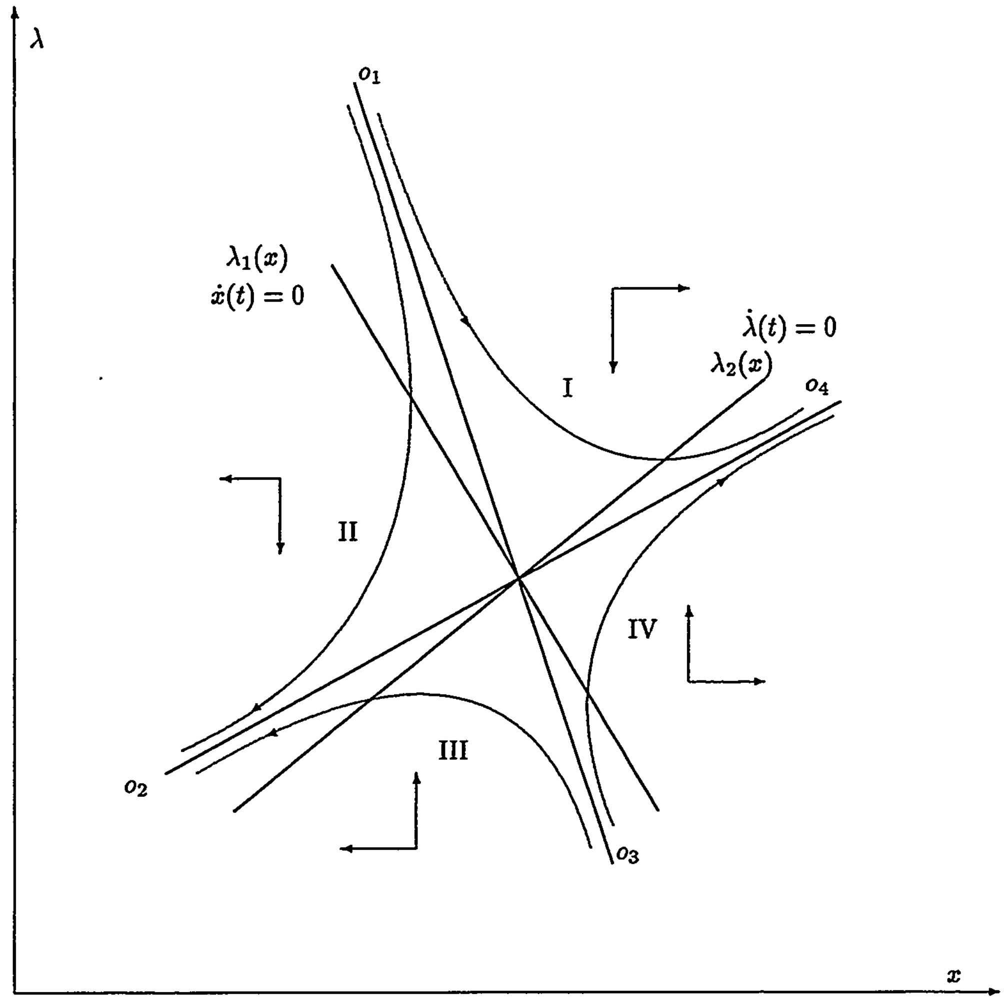

Figure B.1: Evolution of steady state solutions around an equilibrium (a saddle point type) at the state-costate phase plane for the case (2).

The State-Control Phase Plane

Let us consider now the state-control phase plane, i.e. the system (B.11). Now the critical point can be found as an intersection of the isoclines $\dot{x}(t) = 0$ and $\dot{u}(t) = 0$, see Fig. B.2. Let us show that the intersection point is also unique in this case. Let us start with the curve $u_2(x)$ satisfying the equality for $\dot{x}^*(t) = 0$, i.e. $f(x, u_2(x)) = 0$. From the implicit function theorem we have

$$\frac{du_2(x)}{dx} = -\frac{f_x}{f_u} > 0$$

from assumptions (M1) and (M3). Then the function $u_2(x)$ increases with x.

Consider now the curve $u_1(x)$ for $\dot{u}^*(t) = 0$. Here we calculate the derivative only in the equilibrium, i.e. at the point satisfying $\dot{x}^*(t) = 0$ and $\dot{\lambda}^*(t) = 0$. This still needs some more calculations to be done. For $\dot{u}^*(t) = 0$ from (B.10) we have

$$(U_{ux} - U_u \frac{f_{ux}}{f_u})f - U_x f_u - U_u(\delta - f_x) = 0 \tag{B.24}$$

Differentiating now both sides with respect to x (with $u = u_1(x)$), and making use of the condition $\dot{x}^*(t) = f = 0$, after some algebraic manipulations we get

$$\frac{du_1(x)}{dx} = \frac{(U_{ux} - U_u \frac{f_{ux}}{f_u})f_x - U_{xx}f_u - U_x f_{ux} - U_{ux}(\delta - f_x) + U_u f_{xx}}{U_x f_{uu} + U_{uu}(\delta - f_x)} \leq 0 \tag{B.25}$$

Taking now into account the appropriate assumptions (U) and (M) as well as the inequalities (B.23), which are valid because the derivative is in the equilibrium point, we find that the denominator is nonzero (recall (C)) and that the function $u_1(x)$ is either decreasing with x or is constant (horizontal line). In both cases there is one and only one intersection point of the curves $u_1(x)$ and $u_2(x)$. This means that the equilibrium point is unique. To be physically feasible, this point should be in the positive orthant $x > 0, u > 0$.

The isoclines $\dot{x}(t) = 0$ and $\dot{u}(t) = 0$ divide the positive orthant into four isosectors labeled on the figure I, II, III and IV. Let us consider the behaviour of the orbits in different isosectors. Above the curve $\dot{x}(t) = f(x, u) = 0$, for a constant u, we have

$$f \begin{cases} < 0 & \text{for (1)} \\ > 0 & \text{for (2)} \end{cases}$$

because of the appropriate assumptions on f_x. Then in the isosectors I and II a point on the orbit with the growing t moves letfwards for the case (1) and rightwards for the case (2). It is also obvious, that it moves in the opposite directions in the isosectors III and IV.

Let us now consider the orbits above the curve $\dot{u}(t) = 0$. From the next equation after (B.8) we have

$$\dot{u}(t) = -\frac{f_u \dot{\lambda}(t) + (U_{ux} + \lambda(t) f_{ux})\dot{x}(t)}{U_{uu} + \lambda(t) f_{uu}}$$

This equation is more difficult to analyze. However, as we know that the sign of $\dot{u}(t)$ is the same on each side of the curve $\dot{u}(t) = 0$ we consider the line where $\dot{x}(t) = 0$. At this line we have

$$\dot{u}(t)|_{\dot{x}=0} = -\frac{f_u \dot{\lambda}(t)}{U_{uu} + \lambda(t) f_{uu}} \tag{B.26}$$

The denominator is always negative. At the equilibrium point $\dot{\lambda}(t) = 0$. Let us check then only the sign of the numerator partial derivative in u. Denote $l(x, u, \lambda) = -f_u \dot{\lambda}(t) = f_u[U_x - \lambda(\delta - f_x)]$. We have

$$l_u = f_{uu}\dot{\lambda}(t) + f_u[U_{xu} - \lambda_u(\delta - f_x) + \lambda f_{xu}]$$

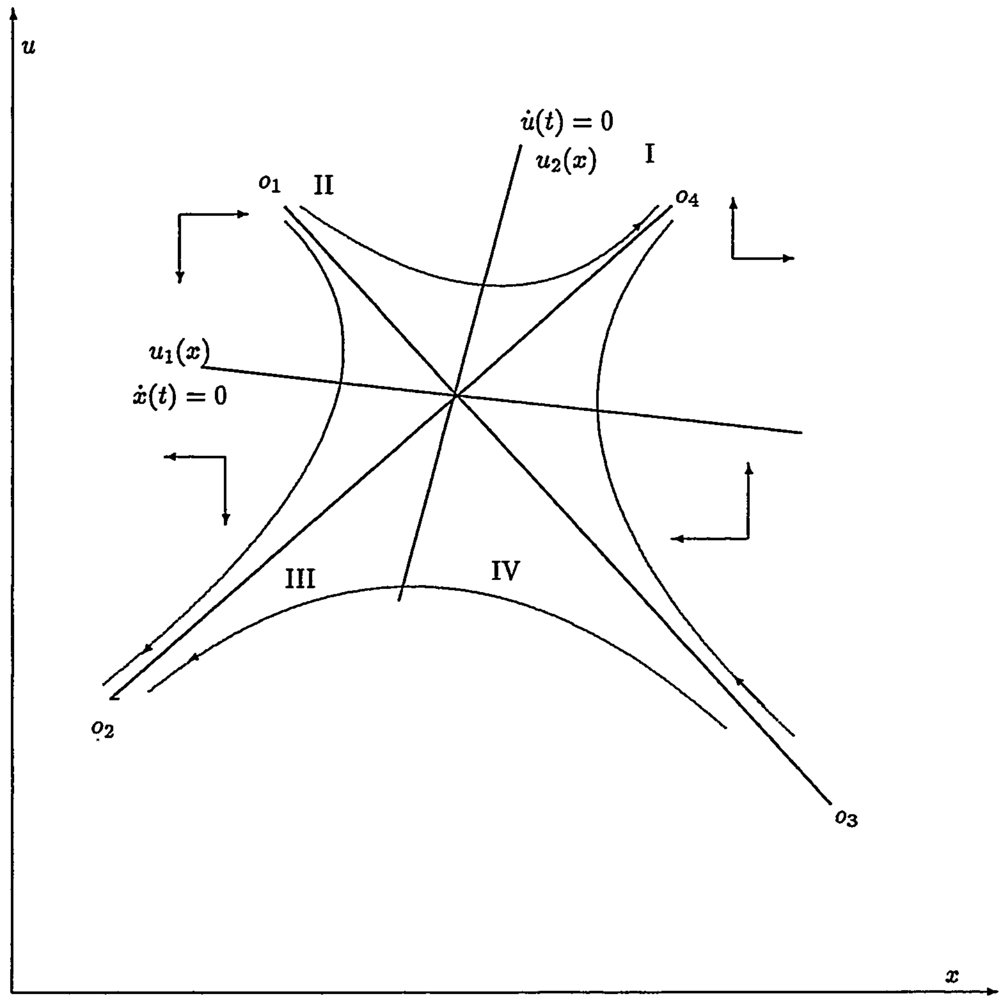

Figure B.2: Evolution of steady state solutions around an equilibrium (a saddle point type) at the state-control phase plane for the case (2).

At the equilibrium the first component on the right hand side is zero. Moreover, from the first equation of (B.7) we get

$$\lambda_u = -\frac{U_{uu} + \lambda f_{uu}}{f_u} \begin{cases} < 0 & \text{for } (1) \\ > 0 & \text{for } (2) \end{cases}$$

Thus at the equilibrium

$$l_u = f_u(U_{xu} + \lambda f_{xu}) + (\delta - f_u)(U_{uu} + \lambda f_{uu})$$

and, taking into accout (B.23)

$$l_u \begin{cases} < 0 & \text{for } (1) \\ > 0 & \text{for } (2) \end{cases}$$

Finally

$$\dot{u}(t) \begin{cases} < 0 & \text{for } (1) \\ > 0 & \text{for } (2) \end{cases}$$

Then above the curve $\dot{u}(t) = 0$ (in the isosectors II and III) the points on the orbits move downwards for the case (1) and upwards for the case (2) and below the curve (in the isosectors I and IV) in the opposite directions. The resulting saddle point neighbourhood is presented on Fig. B.2.

Now, at the equilibrium, from (B.26) the elements h_x and h_u of the Jacobian matrix for the equations (B.11) are

$$h_x = -\frac{f_u \dot{\lambda}_x + (U_{ux} + \lambda(t) f_{ux})\dot{x}_x}{U_{uu} + \lambda(t) f_{uu}}$$

because at the equilibrium $\dot{\lambda}(t) = \dot{x}(t) = 0$. Then we have $\dot{x}_x = f_x$ and from the second and the first of equations (B.7)

$$\dot{\lambda}_x = -(U_{xx} + \lambda(t) f_{xx}) - \frac{1}{f_u}(U_{xu} + \lambda(t) f_{xu})(\delta - f_x)$$

so at the equilibrium

$$h_x = \frac{(U_{xx} + \lambda(t) f_{xx}) f_u + (\delta - 2f_x)(U_{xu} + \lambda(t) f_{xu})}{U_{uu} + \lambda(t) f_{uu}} \begin{cases} > 0 & \text{for } (1) \\ < 0 & \text{for } (2) \end{cases}$$

Similarly

$$h_u = -\frac{f_u \dot{\lambda}_u + (U_{ux} + \lambda(t) f_{ux})\dot{x}_u}{U_{uu} + \lambda(t) f_{uu}}$$

We have $\dot{x}_u = f_u$ and then

$$\dot{\lambda}_u = -(U_{xu} + \lambda(t) f_{xu}) - \frac{1}{f_u}(U_{uu} + \lambda(t) f_{uu})(\delta - f_x)$$

Then

$$h_u = \delta - f_x \begin{cases} > 0 & \text{for } (1) \\ < 0 & \text{for } (2) \end{cases}$$

Finally

$$\det J = \begin{bmatrix} f_x & f_u \\ h_x & h_u \end{bmatrix} = f_x h_u - f_u h_x < 0$$

i.e. the equilibrium is a saddle point.

Let us additionally notice that Feichtinger and Hartl [21] conclude that the Jacobian determinants for both cases (state-costate and state-control) are equal.

B.2 Two or More-Dimensional State

Many results of the previous section extend to more than two differential equations, and some also to the case when functions depend on t. Let x be a n-dimensional vector. Let us again assume the twice continuous differentiability of the functions and consider the set of nonlinear differential equations

$$\dot{x}(t) = F(x(t), t)$$

Let us assume that a steady state solution $x^*(t)$ satisfying $F(x^*(t), t) = 0$ exists. Denote $z(t) = x(t) - x^*(t)$. Then we have

$$\dot{z}(t) = F(x^*(t) + z(t), t) - F(x^*(t), t) = F_x(x^*(t), t)z(t) + f(z, t)$$

where $f(z,t) = o(|z|)$. For the known $x^*(t)$ the derivative $F_x(x^*(t), t)$ is a function of t only and we can always write it in a form $F_x(x^*(t), t) = A + B(t)$, where A does not depend on t. This way we reduce our original problem to the following equation

$$\dot{z}(t) = Az(t) + B(t)z(t) + f(z, t) = Az(t) + g(z, t) \tag{B.27}$$

In the important *autonomous problem* case $F(x(t), t) = F(x(t))$ does not depend explicitly on t. In this case also a steady state solution of the equation $F(x^*) = 0$ does not depend on t and forms a critical point (or an equilibrium). Then from the Taylor expansion

$$\dot{z}(t) = \dot{x}(t) = F_x(x^*)(x(t) - x^*) + O(r^2) = Az(t) + g(z)$$

where $r = |x(t) - x^*|$ ($|\circ|$ is an Euclidean norm) and we easily identify $A = F_x(x^*)$ and $g(z) = O(r^2)$.

Although the definitions of A and $B(t)$ in (B.27) are to some extend arbitrary, only some of them may be significant in further analysis. Namely, we require now, that g be continuous, $g(0,t) = 0$ and that for a given $\epsilon > 0$ there exist δ and t_ϵ that $|g(z,t) - g(\tilde{z},t)| \leq \epsilon|z - \tilde{z}|$ for $|z| \leq \delta, |\tilde{z}| \leq \delta$ and $t > t_\epsilon$ (observe that these conditions are obviously met for the autonomous system considered above). Let all the characteristic roots of A have negative real parts. Then any solution starting close enough to the origin converges to it, i.e. $z^*(t) \to 0$ as $t \to \infty$ (in the autonomous case we can also say that the critical point x^* is stable, of the node type).

Assume now, that k characteristic roots of A have negative real parts and $n - k$ characteristic roots have positive real parts. Then for any sufficiently large t there exist in the z space a real k-dimensional manifold S containing the origin such that any solution $z^*(t)$ starting at the manifold S satisfies $z^*(t) \to 0$ as $t \to 0$. Moreover any solution near the origin but not on S does not converge to it (in the autonomous case the critical point x^* is of the saddle type).

As before, for the autonomous system the solution of a linearized system $\dot{z}(t) = Az(t)$ is of the form

$$z(t) = a_1 W_1 e^{s_1 t} + \ldots + a_k W_k e^{s_k t} + a_{k+1} W_{k+1} e^{s_{k+1} t} + \ldots a_n W_n e^{s_n t}$$

where s_i, $i = 1, 2, \ldots n$ are the eigenvalues and W_i, $i = 1, 2, \ldots n$ are the eigenvectors of the matrix A. So also in this case it is possible to find directions of the convergent manifolds.

However, analysis of the multidimensional equation is much more difficult. Let us consider the case of two-dimensional equation of motion, that is $x(t)$ in (B.2) is now a two dimensional vector $x(t) = [y(t)\ v(t)]^T$ and take for $\dot{x}(t) = F(y, v, u)$ also the following notation

$$\dot{y}(t) = f(y(t), v(t), u(t))$$

$$\dot{v}(t) = g(y(t), v(t), u(t))$$

The maximum principle equations now are

$$U_u + \begin{bmatrix} \eta \\ \lambda \end{bmatrix}^T \begin{bmatrix} f_u \\ g_u \end{bmatrix} = U_u + \eta f_u + \lambda g_u = 0 \tag{B.28}$$

$$\begin{bmatrix} \dot{\eta} \\ \dot{\lambda} \end{bmatrix} - \delta \begin{bmatrix} \eta \\ \lambda \end{bmatrix} = - \begin{bmatrix} U_y \\ U_v \end{bmatrix} - \begin{bmatrix} \eta \\ \lambda \end{bmatrix}^T \begin{bmatrix} f_y & f_v \\ g_y & g_v \end{bmatrix}$$

Then considering the state-costate case we have the following set of four equations to be analyzed (for simplicity dependence on time was dropped)

$$\begin{aligned} \dot{y} &= f(y, v, \hat{u}(y, v)) \\ \dot{v} &= g(y, v, \hat{u}(y, v)) \\ \dot{\eta} &= \delta\eta - U_y(y, v, \hat{u}(y, v)) - \eta f_y(y, v, \hat{u}(y, v)) - \lambda g_y(y, v, \hat{u}(y, v)) \\ \dot{\lambda} &= \delta\lambda - U_v(y, v, \hat{u}(y, v)) - \eta f_v(y, v, \hat{u}(y, v)) - \lambda g_v(y, v, \hat{u}(y, v)) \end{aligned}$$

where $\hat{u}(y, v)$ is an implicit function of the equation (B.28).

Some special cases simplifying the task can be considered. For example, in [21] two such classes of problems are identified. One is when one of the equations, for instance the second, is linear in v, with a coefficient δ, i.e. $g(y, v, u) = \delta v - h(y, u)$ and U and f do not depend on v. Then the second equation becomes $\dot{\lambda}(t) = \delta\lambda(t) - \delta\lambda(t) = 0$ and therefore λ is a constant. Moreover, the equation for v can be solved as $v(t) = \int_0^t e^{-r\tau} h(y, u) d\tau$. This case can then be reduced to the analysis of the one state variable model, see [21].

A second class considered in [21] consists of the models with separable states, in which all second derivatives with respect to states, or states and control, are equal to zero. i.e. $U_{xx} = 0$, $U_{xu} = 0$, $F_{xx} = 0$, $F_{xu} = 0$. This means that first order partial derivatives with respect to states must be some constants independent on neither states nor control. In such a case the costate equations are linear with constant parameters and can be solved. This argument is, of course, true also for more dimensional cases.

The case of full two state variables is analyzed in [21]. It is shown that the eigenvalues $\xi_i, i = 1, 2, 3, 4$ of the linearized system are given by a formula

$$\xi_{1,2,3,4} = \frac{\text{tr } J}{4} \pm \sqrt{(\frac{\text{tr } J}{4})^2 - \frac{K}{2} \pm \frac{1}{2}\sqrt{K^2 - 4 \det J}}$$

where tr stays for the trace of the matrix

$$J = \begin{bmatrix} \dot{y}_y & \dot{y}_v & \dot{y}_\eta & \dot{y}_\lambda \\ \dot{v}_y & \dot{v}_v & \dot{v}_\eta & \dot{v}_\lambda \\ \dot{\eta}_y & \dot{\eta}_v & \dot{\eta}_\eta & \dot{\eta}_\lambda \\ \dot{\lambda}_y & \dot{\lambda}_v & \dot{\lambda}_\eta & \dot{\lambda}_\lambda \end{bmatrix} \tag{B.29}$$

and

$$K = \begin{vmatrix} \dot{y}_y & \dot{y}_\eta \\ \dot{\eta}_y & \dot{\eta}_\eta \end{vmatrix} + \begin{vmatrix} \dot{v}_v & \dot{v}_\lambda \\ \dot{\lambda}_v & \dot{\lambda}_\lambda \end{vmatrix} + 2 \begin{vmatrix} \dot{y}_v & \dot{y}_\lambda \\ \dot{\eta}_v & \dot{\eta}_\lambda \end{vmatrix}$$

Obviously, under the conditions

$$K < 0 \qquad 0 < \det J \leq \frac{K^2}{4}$$

all eigenvalues are real and two of them (connected with y and v, see [21]) are negative, and two other positive. Then under above conditions the problem has a saddle point. In [21] also formulae to calculate the directions of the stable orbits on the plane (y, v) are given.

With full Jacobian matrix calculations for the two state variables are rather cumbersome. But in special cases (as for matrices with many zero entries) calculations may be easier.

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Abstract

The report contains a review of basic models and mathematical tools used in economic regulation problems. It starts with presentation of basic models of capital accumulation, resource depletion, pollution accumulation, and population growth, as well as construction of utility functions. Then the one-state variable model is discussed in details. The basic mathematical methods used consist of application of the maximum principle and phase plane analysis of the differential equations obtained as the necessary conditions of optimality. A summary of basic results connected with these methods is given in appendices.

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